NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

THESIS

SINGLE SOURCE
ERROR ELLIPSE COMBINATION

by

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September, 1996

Thesis Advisor: Vicente Garcia

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REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instruction, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington DC 20503.

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE September 1996	3. REPORT TYPE AND DATES COVERED Master's Thesis

4. TITLE AND SUBTITLE Single Source Error Ellipse Combination

5. FUNDING NUMBERS

6. AUTHOR(S) Joseph R. Orechovesky

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Postgraduate School
Monterey CA 93943-5000

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(maximum 200 words)*

There are a number of military applications in which the geographic location of a signal of interest is of prime importance to the ability of a unit to fulfill its mission. The accuracy of the geographic fix provided to the warfighter can directly affect the success or failure of a mission. One method to improve the accuracy of existing systems is to use the weighted average of a number of intercepts. Each intercept is manifested as an error ellipse comprised of a latitude, longitude, semi-axes, heading and a related Chi-squared distributed probability. Individual error ellipses can be viewed as a quadratic surface perpendicular to the x,y plane of a bivariate normal distribution, the z-axis intersection of which corresponds to a Chi-squared value. By transforming the individual error ellipses to their related location covariance matrices, Gaussian statistics may be used to produce a single location ellipse that combines information from several two-dimensional target location ellipses. By providing a means to fuse data from a given source the warfighter or analyst will be able to more accurately assess a threat and respond.

14. SUBJECT TERMS error, ellipse, geolocation

15. NUMBER OF PAGES 82

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT UL

Approved for public release; distribution is unlimited.

SINGLE SOURCE ERROR
ELLPSE COMBINATION

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Submitted in partial fulfillment
of the requirements for the degree of

MASTER OF SCIENCE IN SYSTEMS TECHNOLOGY
(Space Systems Operations)

from the

NAVAL POSTGRADUATE SCHOOL
September 1996

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ABSTRACT

There are a number of military applications in which the geographic location of a signal of interest is of prime importance to the ability of a unit to fulfill its mission. The accuracy of the geographic fix provided to the warfighter can directly affect the success or failure of a mission. One method to improve the accuracy of existing systems is to use the weighted average of a number of intercepts. Each intercept is manifested as an error ellipse comprised of a latitude, longitude, semi-axes, heading and a related Chi-squared distributed probability. Individual error ellipses can be viewed as a quadratic surface perpendicular to the x,y plane of a bivariate normal distribution, the z-axis intersection of which corresponds to a Chi-squared value. By transforming the individual error ellipses to their related location covariance matrices, Gaussian statistics may be used to produce a single location ellipse that combines information from several two-dimensional target location ellipses. By providing a means to fuse data from a given source the warfighter or analyst will be able to more accurately assess a threat and respond.

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ACKNOWLEDGMENTS

I would like to express my sincerest thanks to all the people that made the completion of this thesis possible. First, I want to express my deepest gratitude to Dr. Gerry Baumgartner for giving me his confidence, enduring patience, guidance and encouragement. Without you this work could not have been possible - thank you. A special thanks to Professor Vicente Garcia, no student could find a more caring or motivating advisor. Finally, I want to say thank you to the many Naval Postgraduate School Faculty and Staff that put on a smile and went out of their way to aid me in one endeavor or another.

I. INTRODUCTION

A. JUSTIFICATION

There are a number of military applications in which the geographic location of a signal of interest is of prime importance to a unit's ability to fulfill its mission. The accuracy of the geographic fix provided to the warfighter can directly affect the success or failure of a mission. It is for this reason we must continually strive to improve our ability to generate the most accurate and condensed geographic solution from all available data.

This thesis describes an algorithm to produce a single location ellipse that is the combination of several two-dimensional target location ellipses. The combination algorithm presented in this document is limited to a stationary single source for data set production. The purpose of this study is to assist the Naval, Command, Control and Ocean Surveillance Center Research, Development, Test and Evaluation Division's (NRaD), Integrated Satellite & Link Communications Division (Code 841) in the documentation of the combination algorithm. This effort is in support of the Classic Crystal project developed by NRaD.

B. SCOPE

The scope of this thesis involves the development of the combination algorithm. A mathematically rigorous presentation of assumptions made in the algorithm and of the supporting theory will be presented. If an appropriate method is available, verification of developed mathematical equations is supplied. It is not within the scope of this project to consider the effect of system dependent biases (e.g. atmospheric delays, refraction) on error ellipses produced by different systems. For this reason the data for this project is limited to error ellipses produced by a single source.

C. CHAPTER OUTLINE

Chapter II presents the underlying statistical theory upon which the Combination Algorithm is based. The chapter culminates with the formula for the optimal estimate of the target location as presented by J.A. Roecker, who based his work on that of Nelson Blachman. The concepts of a bivariate normal distribution and application of Bayes' Theorem are covered as they directly support the formulation of the optimal estimate. Chapter III concerns itself with a number of topics all related to applying the optimal estimate of the target location developed in Chapter II to a real world scenario. Development of the covariance matrices, the effects of adding an observation to an existing solution, ellipse rotation and heading issues are covered in rigorous detail. In Chapter IV the theory and concepts of the previous two chapters are put to use in the presentation of the Combination Algorithm. The process is detailed in a step by step manner that can easily be applied to a number of platforms. Finally, Chapter V offers conclusions and recommendations for further study.

II. STATISTICAL THEORY

A. RELATIONSHIP OF THE ERROR ELLIPSE TO A BIVARIATE NORMAL DISTRIBUTION

Each of the target location error ellipses to be combined to produce a single error ellipse will be defined by the following parameters: center position, heading and magnitude of semi-axes and probability level. These ellipses represent an area that has the specified probability that the target lie within it's bounds. We will begin by defining a joint normal distribution of two variables $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ as

$$\phi(\mathbf{x}) = k * \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \mathbf{B}(\mathbf{x} - \boldsymbol{\mu})\right\} \tag{2.1}$$

where k is determined by the normalization of the total probability to one, $\boldsymbol{\mu}$ and $\mathbf{x}$ are 2-component column vectors and $\mathbf{B}$ a (2×2)-matrix. The vector $\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ gives the location of the center of symmetry of the distribution

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{x} - \boldsymbol{\mu})\phi(\mathbf{x})dx_1 dx_2 = 0 \tag{2.2}$$

The expected or mean value of a random vector $\mathbf{x}$ is

$$\mathrm{E}(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{x}\phi(\mathbf{x})dx_1 dx_2 = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \tag{2.3}$$

The covariance matrix of the distribution is defined as

$$\mathbf{C}(\mathbf{x}) = \mathrm{E}(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} = \left[\mathrm{E}(x_i - \mu_i)(x_j - \mu_j)\right] \tag{2.4}$$

(Anderson,1984). The i th diagonal element of this matrix, $\mathrm{E}(x_i - \mu_i)^2$, is the variance of x_i, and the i, j th off-diagonal element, $\mathrm{E}(x_i - \mu_i)(x_j - \mu_j)$, is the covariance of x_i and x_j, $i \neq j$. Note that since $\mathrm{E}(x_i - \mu_i)(x_j - \mu_j) = \mathrm{E}(x_j - \mu_j)(x_i - \mu_i)$ the covariance matrix is symmetric. With this said the covariance matrix can be written as

$$\mathbf{C} = \mathbf{B}^{-1} = \mathrm{E}\begin{pmatrix} (x_1 - \mu_1)^2 & (x_1 - \mu_1)(x_2 - \mu_2) \\ (x_2 - \mu_2)(x_1 - \mu_1) & (x_2 - \mu_2)^2 \end{pmatrix} \tag{2.5}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \tag{2.6}$$

where we have used the symmetry of the covariance matrix (i.e. $\sigma_{12} = \sigma_{21}$). By matrix inversion we obtain

$$\mathbf{B} = \frac{1}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2}\begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix}. \tag{2.7}$$

To facilitate the next step in our development of the ellipse of covariance we will introduce two reduced variables

$$u_i = \frac{x_i - \mu_i}{\sigma_i}, \quad i = 1,2 \tag{2.8}$$

with the property $\mathrm{var}(u_1) = \mathrm{var}(u_2) = 1$, and the correlation coefficient

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2} = \mathrm{cov}(u_1, u_2). \tag{2.9}$$

Equation 2.1 now takes the reduced form

$$\phi(u_1, u_2) = k * \exp\left(-\frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\mathbf{B}\boldsymbol{u}\right) \tag{2.10}$$

with

$$\mathbf{B} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}. \tag{2.11}$$

Lines of constant probability density are determined by requiring the exponent in Equation 2.10 to be constant, as shown in the following equation:

$$\frac{1}{1-\rho^2}\left(u_1^2 + u_2^2 - 2u_1u_2\rho\right) = c \tag{2.12}$$

If we choose the constant, $c=1$, then in terms of the original variables, Equation 2.12 becomes

$$\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2} - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2} + \frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2} = 1-\rho^2 \tag{2.13}$$

(Brandt, 1983). This can also be written as a sum of squares of two stochastically independent variables

$$\frac{1}{1-\rho^2}\left[\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2} - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2} + \frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right] = \left(\frac{y_1}{\sigma_{y1}}\right)^2 + \left(\frac{y_2}{\sigma_{y2}}\right)^2 = \chi^2 \tag{2.14}$$

which are distributed as χ^2 with two degrees of freedom. This is the equation of an ellipse with the center located at μ_1, μ_2. The semi-axes of the ellipse have an angle θ with respect to the x_1, x_2 axes. This angle, and the magnitude of the semi-major axis **a** and the semi-minor axis **b** can be derived from Equation 2.13 using the properties of conic sections:

$$\tan 2\theta = \frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2} \tag{2.15}$$

$$a^2 = \frac{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}{\sigma_2^2\cos^2\theta - 2\rho_1\sigma_2\sin\theta\cos\theta + \sigma_1^2\sin^2\theta} \tag{2.16}$$

$$b^2 = \frac{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}{\sigma_2^2\sin^2\theta - 2\rho_1\sigma_2\sin\theta\cos\theta + \sigma_1^2\cos^2\theta} \tag{2.17}$$

The ellipse we have just described is known as the ellipse of covariance of the bivariate normal distribution or, in the context of this thesis, the error ellipse. The ellipse will always fall within the rectangle defined by points (μ_1, μ_2) and the standard deviations (σ_1, σ_2). It can been seen from Figure 2.1 that the ellipse will touch the rectangle at four points and in the extreme case of $\rho = \pm 1$ the ellipse will degenerate into a straight line along one of the diagonals of the rectangle. To help visualize the concept, the bivariate normal distribution density function can be thought of as a surface above the plane. The contours of equal density are contours of equal altitude (equal probability) on a topographical map; they indicate the shape of the hill (or probability surface)(Anderson, 1984).

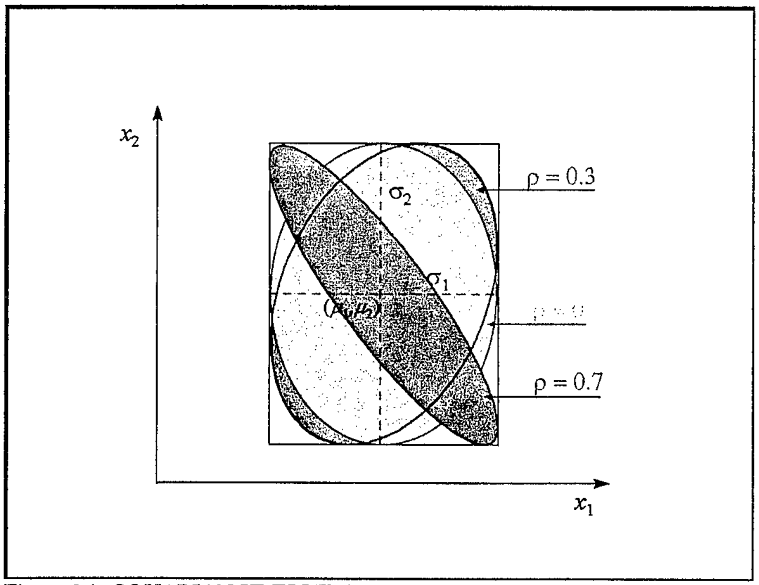


Figure 2.1 COVARIANCE ELLIPSES

The form that the ellipse of covariance was presented in Equation 2.14 did well to illustrate the properties of an error ellipse. In order to implement the combination algorithm we must be able to derive the covariance matrix from the initial parameters. This is most easily accomplished by rewriting Equation 2.14 in matrix form as

$$\chi^2 = \left(\mathbf{x} - \bar{\mathbf{x}}\right)^{\mathrm{T}} \mathbf{C}^{-1}\left(\mathbf{x} - \bar{\mathbf{x}}\right) \tag{2.18}$$

where $\mathbf{x}$ is a Gaussian random vector, $\bar{\mathbf{x}}$ is the true mean of the vector $\mathbf{x}$ and $\mathbf{C}$ is the covariance matrix of $\mathbf{x}$ (Roecker, 91). The chi-squared value χ^2 with 2 degrees of freedom corresponds to the probability that the observation falls within the ellipse described by this equation. In Equations 2.15 thru 2.17 the angle between the ellipse axes and x_1, x_2 axes, and the semi-axes were derived from Equation 2.14. The same information can be extracted from Equation 2.18 with the help of the eigenvalues and eigenvectors of $\mathbf{C}$ (Roecker, 1991). The eigenvalues are found from

$$|\mathbf{C} - \varepsilon \mathbf{I}| = 0 \tag{2.19}$$

where $\mathbf{I}$ is the identity matrix. Substituting for $\mathbf{C}$ from Equation 2.6 and taking the determinant yields

$$\varepsilon^2 - \varepsilon\left(\sigma_1^2 + \sigma_2^2\right) + \sigma_1^2\sigma_2^2 - \sigma_{12}^2 = 0\,. \tag{2.20}$$

Therefore, the eigenvalues are

$$\varepsilon_{\pm} = \frac{\sigma_1^2 + \sigma_2^2 \pm \sqrt{\left(\sigma_1^2 + \sigma_2^2\right)^2 - 4\left(\sigma_1^2\sigma_2^2 - \sigma_{12}^2\right)}}{2} \tag{2.21}$$

and the semi-axes are the square roots of the eigenvalues multiplied by the two-degree of freedom χ^2-value corresponding to the specified probability p

$$a = \sqrt{\chi^2 \varepsilon_+} \tag{2.22}$$

$$b = \sqrt{\chi^2 \varepsilon_-} \tag{2.23}$$

Values of χ^2 vs. p are given in Table 2.1

Table 2.1 Values of Chi-square vs. Probability

χ^2	p
2.30	0.683
4.61	0.90
6.17	0.954
9.21	0.99
11.9	0.9923
18.4	0.9999

B. BAYES' THEORM

This project is intended to combine target error ellipses (discussed in the previous section) generated by several independent observations, each of which can be considered Gaussian. The result of which will be an error ellipse that represents the smallest area containing the target location with the specified probability p. The nucleus of the Combination Algorithm is based on the work of Nelson M. Blachman of GTE Government Systems Corporation. Since Blachman's application of Bayes' Theorem is fundamental to the solution, excerpts of his manuscript (Blachman, 1989) are presented below:

We suppose that, on the basis of I *independent sets of observations* $O_1, O_2, \ldots, O_I$ *of the same target, a location ellipse has been found. Each such ellipse is a contour of the Gaussian conditional probability density function (pdf)* $p(x, y|O_i)$ *of the target location based on one set* O_i *of observations. Each of the* I *given ellipses is based on the Bayesian relationship*

$$p\left(x,y\middle|O_i\right)=\frac{p_i(x,y)p\left(O_i\middle|x,y\right)}{p(O_i)} \tag{2.24}$$

expressing the probability density function (pdf) of the target (x,y) *conditioned on the observations* O_i *as the ratio of the product of the prior (i.e. before* O_i*) pdf of* (x,y) *times the pdf of* O_i *conditioned on the location* (x,y) *(this product being the joint pdf of* (x,y) *and* O_i*) to a normalizing constant, which is the unconditioned pdf of* O_i *and is the integral of the foregoing product over all* x *and* y*. Because different observers may utilize different prior pdfs,* $p_i(x,y)$ *can depend on i. As long as* $p_i(x,y)$ *is constant over the entire area where* $p\left(O_i\middle|x,y\right)\neq 0$*, it cancels out here because* $p(O_i)$ *is proportional to it, and so its value is not important.*

For each I *the pdf of* O_i *conditioned on target location* (x,y) *is therefore*

$$p\left(O_i\middle|x,y\right)=\frac{p(O_i)p\left(x,y\middle|O_i\right)}{p_i(x,y)} \tag{2.25}$$

On account of the statistical independence of the different sets of observations for a given target location, the product of $p\left(O_i\middle|x,y\right)$ *over all* i *is the conditional pdf* $p\left(O_1,O_2,\ldots,O_I\middle|x,y\right)$ *of the entire set of observations for a given target location* (x,y)*. Multiplying it by the prior pdf* $p(x,y)$*, we get the joint pdf* $p(O_1,O_2,\ldots,O_I,x,y)$ *of the observations and the target location. Dividing by* $p(O_1,O_2,\ldots,O_I)$*, which is the integral of the latter over all* x *and* y*, we get, by Bayes' theorem, the pdf of* (x,y) *conditioned on* all *observations,*

$$p(x,y|O_1,\ldots,O_I) = \frac{p(x,y)}{p(O_1,\ldots,O_I)} \times \prod_{i=1}^{I} \frac{p(O_i)p(x,y|O_i)}{p_i(x,y)} \qquad (2.26)$$

The factors $p(O_i)$ and $p(O_1,\ldots,O_I)$, which depend only on the observations, serve merely to normalize this posterior pdf; they do not effect its shape. As long as the prior pdf $p(x,y)$ is constant over the whole area where the target may lie, it cancels out as before, and it too does not affect the shape of $p(x,y|O_1,\ldots,O_I)$, which is thus determined as Equation 2.26 indicates, by the product of the pdfs $p(x,y|O_i)$ based on the individual sets of observations.

Blachman has set the theoretical foundation from which a practical combination algorithm can be developed. In order to apply Equation 2.26 to a spherical earth we must have the ability to manipulate the observations by a constant scaling factor. Roecker while expanding on Blachman's work (Roecker, 1991) noted that,

as long as the prior pdfs $p(X)$ and $p_i(X)$ are constant over the areas of interest, they will cancel out with $p(O_1,\ldots,O_I)$ and $p(O_i)$ to affect $p(X|O_1,\ldots,O_I)$ only as a scaling factor. This leaves the conditional pdf $p(X|O_1,\ldots,O_I)$ completely dependent on the product of $p(X|O_i)$ for its shape, and can be expressed as

$$p(X|O_1,\ldots,O_I) = K\prod_{i=1}^{I} p(X|O_i) \qquad (2.27)$$

where K is the scaling factor and X is a vector containing the target location. This observation will later allow the observation covariance matrices to be scaled, compensating for a spherical earth.

C. METHODOLOGY

In order to apply Equation 2.27 a method of extracting an optimal estimate of the target location must be developed. Since $p(\mathbf{x}|\mathbf{o}_i)$ is Gaussian, the conditional pdf of the target location can be written as

$$p(\mathbf{x}|\mathbf{o}_1,\ldots,\mathbf{o}_M) = K_1 \prod_{i=1}^{M} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{o}_i)^{\mathrm{T}} \mathbf{C}_i^{-1} (\mathbf{x}-\mathbf{o}_i)\right\} \tag{2.28}$$

$$p(\mathbf{x}|\mathbf{o}_1,\ldots,\mathbf{o}_M) = K_1 \exp\left\{-\frac{1}{2}\sum_{i=1}^{M}(\mathbf{x}-\mathbf{o}_i)^{\mathrm{T}} \mathbf{C}_i^{-1} (\mathbf{x}-\mathbf{o}_i)\right\} \tag{2.29}$$

where K_1 is the scaling factor, $\mathbf{x}$ the vector containing the target location, $\mathbf{o}_i$ the i th observation of $\mathbf{x}$ and $\mathbf{C}$ the covariance matrix of $\mathbf{x}$ (Roecker, 1991). The optimal estimate for any symmetric cost function is the same for Gaussian distributions (Van Trees, 1968) and results by minimizing the pdf detailed in Equation 2.29. The minimum of Equation 2.29 is found by setting the first derivative to zero as presented below

$$\frac{\partial p}{\partial \mathbf{x}} = K_1 e^{\{E_M\}} \left(-\frac{1}{2}\frac{\partial E_M}{\partial \mathbf{x}}\right) \tag{2.30}$$

where E_M is defined as

$$E_M = \sum_{i=1}^{M}(\mathbf{x}-\mathbf{o}_i)^{\mathrm{T}} \mathbf{C}_i^{-1} (\mathbf{x}-\mathbf{o}_i) \tag{2.31}$$

and where

$$\frac{\partial E_M}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial E_M}{\partial x_1} \\ \vdots \\ \frac{\partial E_M}{\partial x_n} \end{pmatrix} \tag{2.32}$$

Note that M is the number of observations and n is the number of components of the vectors $\mathbf{x}$ and $\mathbf{o}$ (n=2 for this problem). To begin with we will examine the first observation $\mathbf{o}_1$

$$E_1 = (\mathbf{x} - \mathbf{o}_1)^{\mathrm{T}} \mathbf{C}_1^{-1} (\mathbf{x} - \mathbf{o}_1) \tag{2.33}$$

$$= \begin{bmatrix} x_1 - o_{11} & \cdots & x_n - o_{1n} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \\ B_{n1} & & B_{nn} \end{bmatrix} \begin{bmatrix} x_1 - o_{11} \\ \vdots \\ x_n - o_{1n} \end{bmatrix} \tag{2.34}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} B_{i1}(x_i - o_{1i}) & \cdots & \sum_{i=1}^{n} B_{in}(x_i - o_{1i}) \end{bmatrix} \begin{bmatrix} x_1 - o_{11} \\ \vdots \\ x_n - o_{1n} \end{bmatrix} \tag{2.35}$$

$$= \sum_{j=1}^{n} (x_j - o_{1j}) \sum_{i=1}^{n} B_{ij} (x_i - o_{1i}) \tag{2.36}$$

then by the product rule we have

$$\frac{\partial E_1}{\partial x_k} = \sum_{i=1}^{n} B_{ik}(x_i - o_{1i}) + \sum_{j=1}^{n} (x_j - o_{1j}) B_{kj} \tag{2.37}$$

since all summing is to n we can make a change of indices for readability

$$\frac{\partial E_1}{\partial x_k} = \sum_{i=1}^{n} B_{ik}(x_i - o_{1i}) + \sum_{i=1}^{n} B_{ki}(x_i - o_{1i}) \tag{2.38}$$

Because the covariance matrix is symmetric we have

$$\frac{\partial E_1}{\partial x_k} = \sum_{i=1}^{n} B_{ik}\left(x_i - o_{1i}\right) + \sum_{i=1}^{n} B_{ki}\left(x_i - o_{1i}\right) \tag{2.39}$$

$$= 2\sum_{i=1}^{n} B_{ki}\left(x_i - o_{1i}\right) \tag{2.40}$$

In matrix notation this becomes

$$\frac{\partial E_1}{\partial \mathbf{x}} = 2\mathbf{C}_1^{-1}\left(\mathbf{x} - \mathbf{o}_1\right) \tag{2.41}$$

Setting the derivative $\dfrac{\partial E}{\partial \mathbf{x}}$ to zero we have

$$2\mathbf{C}_1^{-1}\left(\mathbf{x} - \mathbf{o}_1\right) = 0 \tag{2.42}$$

$$\mathbf{C}_1^{-1}\mathbf{x} - \mathbf{C}_1^{-1}\mathbf{o}_1 = 0 \tag{2.43}$$

$$\mathbf{x} = \mathbf{o}_1 \tag{2.44}$$

So the initial estimate for $\mathbf{x}$ is the first observation $\mathbf{o}_1$, which was to be expected. Now that the process for a single observation has been established, we can generalize to multiple observations. To begin with we expand the reduction variable E to include M observations

$$E_M = \sum_{m=1}^{M}\left(\sum_{j=1}^{n}\left(x_j - o_{mj}\right)\sum_{i=1}^{n} A_{ij}\left(x_i - o_{mi}\right)\right) \tag{2.45}$$

$$= \sum_{m=1}^{M}\left(\mathbf{x} - \mathbf{o}_m\right)^{\mathrm{T}}\mathbf{C}_m^{-1}\left(\mathbf{x} - \mathbf{o}_m\right) \tag{2.46}$$

and once again set the derivative to zero

$$\frac{\partial E_M}{\partial \mathbf{x}} = 2\sum_{m=1}^{M} \mathbf{C}_m^{-1}\left(\overline{\mathbf{x}} - \mathbf{o}_m\right) = 0 \tag{2.47}$$

$$\sum_{m=1}^{M}\left[\mathbf{C}_m^{-1}\left(\overline{\mathbf{x}} - \mathbf{o}_m\right)\right] = 0 \tag{2.48}$$

$$\left(\sum_{m=1}^{M}\mathbf{C}_m^{-1}\right)\overline{\mathbf{x}} - \sum_{m=1}^{M}\left(\mathbf{C}_m^{-1}\mathbf{o}_m\right) = 0 \tag{2.49}$$

$$\mathbf{C}^{-1}\overline{\mathbf{x}} - \sum_{m=1}^{M}\mathbf{C}\left(\mathbf{C}_m^{-1}\mathbf{o}_m\right) = 0 \tag{2.50}$$

Which, with a little algebraic manipulation yields

$$\overline{\mathbf{x}} = \mathbf{C}\sum_{m=1}^{M}\left(\mathbf{C}_m^{-1}\mathbf{o}_m\right) \quad \text{where} \quad \mathbf{C} = \left(\sum_{m=1}^{M}\mathbf{C}_m^{-1}\right)^{-1} \tag{2.51}$$

As depicted in Figure 2.2, $\overline{\mathbf{x}}$ is the vector representing the location of the weighted average of M number of o observations and $\mathbf{C}$ is the resultant covariance matrix. This is the optimal estimate of the target location we desire and the premise upon which the Combination Algorithm is based.

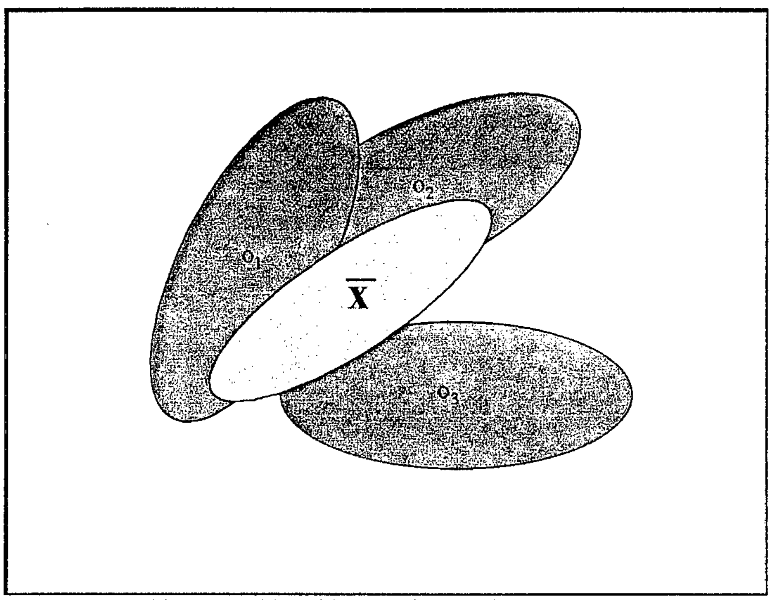


Figure 2.2 OPTIMAL ESTIMATE OF TARGET LOCATION

III. CONCEPT DEVELOPMENT

A. COVARIANCE MATRICES

1. Scaled Covariance Matrix

Chapter II laid the theoretical groundwork for the Combination Algorithm. We have seen that the target error ellipses we are attempting to combine can be modeled by an ellipse of covariance. The covariance matrix being the structure defining the target error ellipse. In the real world problem, the error ellipse will have been generated upon the spherical earth. In order to reduce the overall problem to a two dimensional one, each ellipse will be projected into a plane. Therefore, the covariance matrices will need to be modified, compensating for this projection from the spherical earth to the plane. This modification is to compensate for the inequality between latitude error and longitude error as the latitude increases from zero. The geometry of the projection from a spherical earth to a tangent plane is depicted in Figure 3.1. We need to find the distance a_{plane} (length of projected ellipse axes) when given the arc-length along the surface of the sphere a_{sphere} (an ellipse axes), where r_e = radius of earth and

$$a_{sphere} = r_e \theta \tag{3.1}$$

$$\theta = \frac{a_{sphere}}{r_e}. \tag{3.2}$$

Observing the geometry

$$\tan\theta = \frac{a_{plane}}{r_e} \tag{3.3}$$

$$a_{plane} = r_e \tan\theta \tag{3.4}$$

which leads to the transformation

$$a_{plane} = r_e \tan\left(\frac{a_{sphere}}{r_e}\right) \tag{3.5}$$

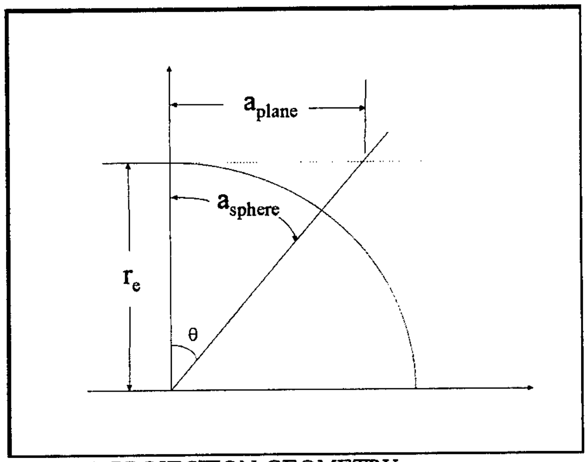


Figure 3.1 PROJECTION GEOMETRY

The congruence transformation

$$\tilde{\mathbf{C}} = \mathbf{S}^T \mathbf{C} \mathbf{S} \tag{3.6}$$

produces the scaled covariance matrix $\tilde{\mathbf{C}}$ that has maintained the symmetry of the original covariance matrix.. The scaling matrix $\mathbf{S}$ is obtained by observing from Figure 3.2 below that the arc-length measured along the surface of the earth at constant latitude ϕ corresponding to a change in longitude $\Delta\lambda$ is given by $r_e \cos(\phi)\Delta\lambda$; and the arc-length measured along the surface of the earth at constant longitude λ corresponding to a change in latitude $\Delta\phi$ is given by $r_e\Delta\phi$. To put latitude and longitude on equal footing (in terms of arc-length) changes in latitude are multiplied by r_e and changes in longitude are multiplied by $r_e \cos(\phi)$. Therefore,

$$\mathbf{S} = \begin{pmatrix} r_e & 0 \\ 0 & r_e \cos\phi \end{pmatrix} \tag{3.7}$$

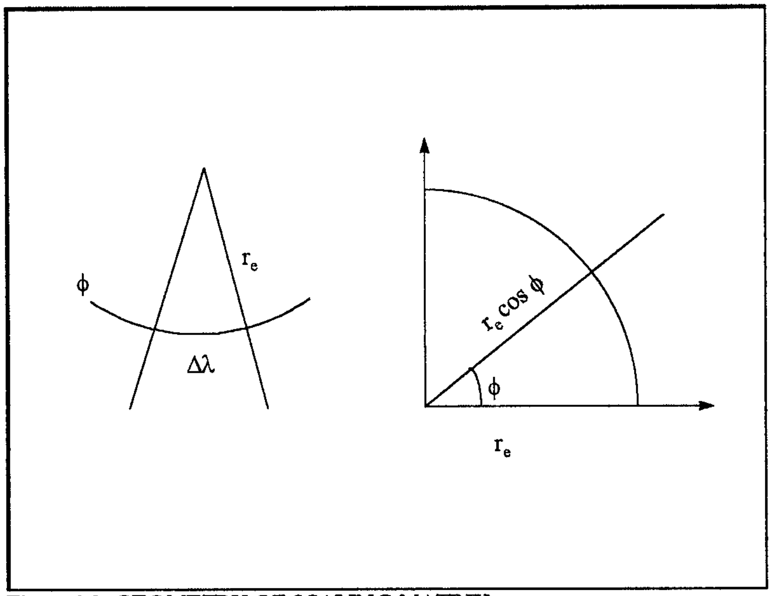


Figure 3.2 GEOMETRY OF SCALING MATRIX

Expanding Equation 3.6 we find that

$$\tilde{\mathbf{C}} = \begin{pmatrix} r_e & 0 \\ 0 & r_e \cos\phi \end{pmatrix} \begin{pmatrix} \sigma_{\phi\phi} & \sigma_{\phi\lambda} \\ \sigma_{\lambda\phi} & \sigma_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} r_e & 0 \\ 0 & r_e \cos\phi \end{pmatrix} \tag{3.8}$$

$$= \begin{pmatrix} r_e \sigma_{\phi\phi} & r_e \sigma_{\phi\lambda} \\ r_e \sigma_{\lambda\phi} \cos\phi & r_e \sigma_{\lambda\lambda} \cos\phi \end{pmatrix} \begin{pmatrix} r_e & 0 \\ 0 & r_e \cos\phi \end{pmatrix} \tag{3.9}$$

$$= \begin{pmatrix} r_e^2 \sigma_{\phi\phi} & r_e^2 \sigma_{\phi\lambda} \cos\phi \\ r_e^2 \sigma_{\lambda\phi} \cos\phi & r_e^2 \sigma_{\lambda\lambda} \cos^2 \phi \end{pmatrix} \tag{3.10}$$

This is the scaled covariance matrix, where ϕ = latitude, λ = longitude and r_e is the radius of the earth. In the section to follow it will be necessary to solve for the eigenvectors of this matrix. To aid in that process, the following notation will be used to refer to the scaled covariance matrix in Equation 3.10.

$$\tilde{\mathbf{C}} = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \tag{3.11}$$

where use has been made of the fact that $\sigma_{\phi\lambda} = \sigma_{\lambda\phi}$.

2. 2x2 to 3x3 Covariance Matrix Conversion

The objective is to convert the 2x2 covariance matrix given in terms of latitude and longitude (Equation 3.11) to a 3x3 covariance matrix in terms of (x,y,z). The transformation is as follows

$$\mathbf{C}_{(x,y,z)} = \mathbf{T}\,\mathbf{C}_{(\phi,\lambda,h)}\mathbf{T}^{\mathrm{T}} \tag{3.12}$$

where the 3x3 covariance matrices are

$$\mathbf{C}_{(x,y,z)} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \tag{3.13}$$

$$\mathbf{C}_{(\phi,\lambda,h)} = \begin{pmatrix} \sigma_{\phi\phi} & \sigma_{\phi\lambda} & \sigma_{\phi h} \\ \sigma_{\lambda\phi} & \sigma_{\lambda\lambda} & \sigma_{\lambda h} \\ \sigma_{h\phi} & \sigma_{h\lambda} & \sigma_{hh} \end{pmatrix} \tag{3.14}$$

and the 3x3 transformation matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{pmatrix} \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \lambda} & \dfrac{\partial x}{\partial h} \\ \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \lambda} & \dfrac{\partial y}{\partial h} \\ \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \lambda} & \dfrac{\partial z}{\partial h} \end{pmatrix} \tag{3.15}$$

where for a spherical earth

$$x = \left(r_e + h\right)\cos\phi \; \cos\lambda \tag{3.16}$$

$$y = \left(r_e + h\right)\cos\phi \; \sin\lambda \tag{3.17}$$

$$z = \left(r_e + h\right)\sin\phi \tag{3.18}$$

r_e being the radius of the earth and the partial derivatives are

$$\frac{\partial x}{\partial \phi} = -(r_e + h)\,\sin(\phi)\cos(\lambda) \tag{3.19}$$

$$\frac{\partial x}{\partial \lambda} = -(r_e + h)\,\cos(\phi)\sin(\lambda) \tag{3.20}$$

$$\frac{\partial x}{\partial h} = \cos(\phi)\cos(\lambda) \tag{3.21}$$

$$\frac{\partial y}{\partial \phi} = -(r_e + h)\,\sin(\phi)\sin(\lambda) \tag{3.22}$$

$$\frac{\partial y}{\partial \lambda} = (r_e + h)\,\cos(\phi)\cos(\lambda) \tag{3.23}$$

$$\frac{\partial y}{\partial h} = \cos(\phi)\sin(\lambda) \tag{3.24}$$

$$\frac{\partial z}{\partial \phi} = (r_e + h)\,\cos(\phi) \tag{3.25}$$

$$\frac{\partial z}{\partial \lambda} = 0 \tag{3.26}$$

$$\frac{\partial z}{\partial h} = \sin(\phi)\,, \tag{3.27}$$

To define $\mathbf{C}_{(\phi,\lambda,h)}$ we start by finding the unscaled covariance matrix $\mathbf{C}_{(\phi,\lambda)}$, from Equation 3.6

$$\mathbf{C}_{(\phi,\lambda)} = \left(\mathbf{S}^{-1}\right)^{\mathrm{T}} \widetilde{\mathbf{C}}\, \mathbf{S}^{-1} \tag{3.28}$$

where

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{r_e} & 0 \\ 0 & \frac{1}{r_e \cos\phi} \end{pmatrix} \tag{3.29}$$

and the scaled covariance matrix $\tilde{\mathbf{C}}$ is given in Equation 3.11. Using the unscaled covariance matrix in terms of latitude and longitude $\mathbf{C}_{(\phi,\lambda)}$ and assuming the height h to be zero with zero uncertainty, we can form the matrix

$$\mathbf{C}_{(\phi,\lambda,h)} = \begin{pmatrix} \sigma_{\phi\phi} & \sigma_{\phi\lambda} & 0 \\ \sigma_{\phi\lambda} & \sigma_{\lambda\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.30}$$

3. Selection of Eigenvalues and Eigenvectors

The key factors in the transformation of an observation seen in terms of semi-axes a, b and heading θ, to it's description by a covariance matrix, are it's eigenvalues and eigenvectors. This section will accomplish four tasks: derive the eigenvalues from the characteristic polynomial of Equation 3.11, derive various identities from the characteristic polynomial that will be useful in later sections, normalize and solve for the eigenvectors, and determine which of the solutions will be numerically stable.

a. *Eigenvalues*

In order to produce the eigenvalues, which will be denoted by ε, the nonlinear equation

$$(\tilde{\mathbf{C}} - \varepsilon\,\mathbf{I})\mathbf{x} = 0 \tag{3.31}$$

needs to be solved. To begin the process the scaled covariance matrix $\tilde{\mathbf{C}}$ is shifted by $\varepsilon\mathbf{I}$, where $\mathbf{I}$ signifies the identity matrix

$$\tilde{\mathbf{C}} - \varepsilon\,\mathbf{I} = \begin{pmatrix} \alpha - \varepsilon & \gamma \\ \gamma & \beta - \varepsilon \end{pmatrix} \tag{3.32}$$

By definition (Strang, 1988) the number ε is an eigenvalue of $\tilde{\mathbf{C}}$ if and only if $\det\left(\tilde{\mathbf{C}} - \varepsilon\,\mathbf{I}\right) = 0$. This determinant leads to the *characteristic polynomial.* It's roots, i.e. where the determinant is zero, are the eigenvalues.

$$\det\left(\tilde{\mathbf{C}} - \varepsilon\,\mathbf{I}\right) = (\alpha - \varepsilon)(\beta - \varepsilon) - \gamma^2 \tag{3.33}$$

$$= \varepsilon^2 - (\alpha + \beta)\varepsilon + \alpha\beta - \gamma^2 = 0 \tag{3.34}$$

The real solutions of the characteristic polynomial (the eigenvalues) are found by the quadratic formula as show below.

$$\varepsilon_{\pm} = \frac{(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta + 4\gamma^2}}{2} \tag{3.35}$$

$$= \frac{(\alpha + \beta) \pm \sqrt{\alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta + 4\gamma^2}}{2} \tag{3.36}$$

$$= \frac{(\alpha + \beta) \pm \sqrt{\alpha^2 - 2\alpha\beta + \beta^2 + 4\gamma^2}}{2} \tag{3.37}$$

$$= \frac{(\alpha + \beta) \pm \sqrt{(\alpha - \beta)^2 + 4\gamma^2}}{2} \tag{3.38}$$

b. Identities

With the characteristic polynomial at hand, this is a good time to observe two identities that will become useful in future sections. Both of the following identities are obtained from factorization of the general quadratic equation, which is

$$\left(\varepsilon-\varepsilon_{+}\right)\left(\varepsilon-\varepsilon_{-}\right)=0 \tag{3.39}$$

$$\varepsilon^{2}-\left(\varepsilon_{+}+\varepsilon_{-}\right)\varepsilon+\varepsilon_{+}\varepsilon_{-}=0\,. \tag{3.40}$$

Comparing Equation 3.40 with the characteristic polynomial Equation 3.34 we find two identities that will later aid in establishing relationships between the scaled covariance matrix Equation 3.11 and the primary data parameters $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{\theta}$,

$$\varepsilon_{+}+\varepsilon_{-}=\alpha+\beta \tag{3.41}$$

$$\varepsilon_{+}\varepsilon_{-}=\alpha\beta-\gamma^{2} \tag{3.42}$$

Also worth noting, but of less significant impact, are the identities

$$\varepsilon_{+}-\varepsilon_{-}=\sqrt{(\alpha-\beta)^{2}+4\gamma^{2}}\,, \tag{3.43}$$

$$\varepsilon_{+}\Big|_{\gamma=0}=\alpha \quad \text{and} \quad \varepsilon_{-}\Big|_{\gamma=0}=\beta\,. \tag{3.44}$$

c. *Eigenvectors*

This section will present solutions for the eigenvectors $\mathbf{x}_i$ where $\mathbf{x}_i=\begin{pmatrix}x_{1i}\\x_{2i}\end{pmatrix}, i=\pm$. Writing Equation 3.31 in terms of Equation 3.32 and the notation just mentioned yields

$$\begin{pmatrix}\alpha-\varepsilon & \gamma\\ \gamma & \beta-\varepsilon\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{3.45}$$

Equation 3.45 represents a system of two equations with two unknowns. By adding the formula

$$x_1^2+x_2^2=1 \tag{3.46}$$

to the system of equations, we can simultaneously normalizing the eigenvector to unit length and supply an additional formula that will be used to produce four valid normalized solutions. Each pair of equations will generate solutions for the positive and negative roots of Equation 3.38. Each solution will be checked for computational stability. The most stable vector from each root will eventually be used to populate the eigenvector matrix.

(1) Solution for $\mathbf{x}_+^{(1)}$. The first solution presented is for $\mathbf{x}_+^{(1)}$, which corresponds to the solution of the first equation in Equation 3.45. When $\varepsilon = \varepsilon_+$

$$(\alpha - \varepsilon_+)x_1 + \gamma x_2 = 0 \tag{3.47}$$

$$x_1 = -\frac{\gamma}{\alpha - \varepsilon_+}x_2 \tag{3.48}$$

The second equation needed to solve this system is supplied by normalizing the eigenvectors to unit length using Equation 3.46. By substitution we have

$$\left[\frac{\gamma^2}{(\alpha - \varepsilon_+)^2} + 1\right]x_2^2 = 1 \tag{3.49}$$

Solving for x_2 we get

$$x_2^2 = \frac{(\alpha - \varepsilon_+)^2}{(\alpha - \varepsilon_+)^2 + \gamma^2} \tag{3.50}$$

$$x_2 = \frac{(\alpha - \varepsilon_+)}{\sqrt{(\alpha - \varepsilon_+)^2 + \gamma^2}} \tag{3.51}$$

Substituting Equation 3.51 back into Equation 3.46 yields

$$x_1 = \frac{-\gamma}{\sqrt{(\alpha - \varepsilon_+)^2 + \gamma^2}} \tag{3.52}$$

Therefore,

$$\mathbf{x}_{+}^{(1)} = \frac{1}{\sqrt{(\alpha-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix} -\gamma \\ \alpha-\varepsilon_{+} \end{pmatrix} \tag{3.53}$$

Check:

$$\tilde{\mathbf{C}}\mathbf{x}_{+}^{(1)} = \frac{1}{\sqrt{(\alpha-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}\begin{pmatrix} -\gamma \\ \alpha-\varepsilon_{+} \end{pmatrix} \tag{3.54}$$

$$= \frac{1}{\sqrt{(\alpha-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix} -\alpha\gamma+\alpha\gamma-\gamma\varepsilon_{+} \\ -\gamma^{2}+\alpha\beta-\beta\varepsilon_{+} \end{pmatrix} \tag{3.55}$$

$$= \frac{1}{\sqrt{(\alpha-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix} -\gamma\varepsilon_{+} \\ \varepsilon_{+}\varepsilon_{-}-\beta\varepsilon_{+} \end{pmatrix} \tag{3.56}$$

$$= \frac{\varepsilon_{+}}{\sqrt{(\alpha-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix} -\gamma \\ \varepsilon_{-}-\beta \end{pmatrix} \tag{3.57}$$

Now Equation 3.41 states

$$\varepsilon_{-}-\beta = \alpha-\varepsilon_{+} \ ,$$

therefore,

$$\tilde{\mathbf{C}}\mathbf{x}_{+}^{(1)} = \varepsilon_{+}\mathbf{x}_{+}^{(1)}$$

and Equation 3.31 holds true.

In checking for numerical stability note that when $\gamma \to \mathbf{0}$ we obtain $\frac{0}{0}$ for the components of $\mathbf{x}_{+}^{(1)}$, therefore $\mathbf{x}_{+}^{(1)}$ is not numerically stable as $\gamma \to \mathbf{0}$.

(2) Solution for $\mathbf{x}_+^{(2)}$. The second solution evaluated is $\mathbf{x}_+^{(2)}$, which corresponds to the solution of the second equation in Equation3.45 when $\varepsilon = \varepsilon_+$

$$\gamma x_1 + (\beta - \varepsilon_+) x_2 = 0 \tag{3.58}$$

$$x_1 = -\frac{\beta - \varepsilon_+}{\gamma} x_2 \tag{3.59}$$

The second equation needed to solve this system, as in the previous solution, is supplied by normalizing the eigenvectors to unit length using Equation 3.46. By substitution we see

$$\left[\frac{(\beta - \varepsilon_+)^2}{\gamma^2} + 1\right] x_2^2 = 1 \tag{3.60}$$

solving for x_2 we get

$$x_2^2 = \frac{\gamma^2}{(\beta - \varepsilon_+)^2 + \gamma^2} \tag{3.61}$$

$$x_2 = \frac{\gamma}{\sqrt{(\beta - \varepsilon_+)^2 + \gamma^2}} \tag{3.62}$$

Substituting Equation 3.62 back into Equation 3.46 yields

$$x_1 = \frac{\varepsilon_+ - \beta}{\sqrt{(\beta - \varepsilon_+)^2 + \gamma^2}} \tag{3.63}$$

Therefore

$$\mathbf{x}_+^{(2)} = \frac{1}{\sqrt{(\beta - \varepsilon_+)^2 + \gamma^2}} \begin{pmatrix} \varepsilon_+ - \beta \\ \gamma \end{pmatrix} \tag{3.64}$$

Check:

$$\tilde{\mathbf{C}}\mathbf{x}_{+}^{(2)}=\frac{1}{\sqrt{(\beta-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha & \gamma\\ \gamma & \beta\end{pmatrix}\begin{pmatrix}\varepsilon_{+}-\beta\\ \gamma\end{pmatrix} \tag{3.65}$$

$$=\frac{1}{\sqrt{(\beta-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha\varepsilon_{+}-\alpha\beta+\gamma^{2}\\ \gamma\varepsilon_{+}-\beta\gamma+\beta\gamma\end{pmatrix} \tag{3.66}$$

$$=\frac{1}{\sqrt{(\beta-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha\varepsilon_{+}-\varepsilon_{+}\varepsilon_{-}\\ \gamma\varepsilon_{+}\end{pmatrix} \tag{3.67}$$

$$=\frac{\varepsilon_{+}}{\sqrt{(\beta-\varepsilon_{+})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha-\varepsilon_{-}\\ \gamma\end{pmatrix} \tag{3.68}$$

Using the Equation 3.41

$$\alpha-\varepsilon_{-}=\varepsilon_{+}-\beta$$

yields

$$\tilde{\mathbf{C}}\mathbf{x}_{+}^{(2)}=\varepsilon_{+}\mathbf{x}_{+}^{(2)}$$

and Equation 3.31 holds true.

In checking for numerical stability we note that when $\gamma\to\mathbf{0}$

$$\mathbf{x}_{+}^{(2)}\to\frac{1}{\alpha-\beta}\begin{pmatrix}-(\beta-\alpha)\\ 0\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.69}$$

Hence, $\mathbf{x}_{+}^{(2)}$ is a numerically stable solution. Therefore we will use Equation 3.64 for the solution from the eigenvalue ε_{+}. The next step is to determine which of the two equations from the eigenvalue ε_{-}. will be used.

(3) Solution for $\mathbf{x}_-^{(1)}$. The next solution presented is for $\mathbf{x}_-^{(1)}$, which corresponds to the solution of the first equation of Equation 3.45 with $\varepsilon = \varepsilon_-$

$$(\alpha - \varepsilon_-)x_1 + \gamma x_2 = 0 \tag{3.70}$$

$$x_1 = -\frac{\gamma}{\alpha - \varepsilon_-} x_2 \tag{3.71}$$

The second equation needed to solve this system of equations is supplied by normalizing the eigenvectors to unit length using Equation 3.46. By substitution we see

$$\left[\frac{\gamma^2}{(\alpha - \varepsilon_-)^2} + 1\right] x_2^2 = 1 \tag{3.72}$$

Solving for x_2 we get

$$x_2^2 = \frac{(\alpha - \varepsilon_-)^2}{(\alpha - \varepsilon_-)^2 + \gamma^2} \tag{3.73}$$

$$x_2 = \frac{(\alpha - \varepsilon_-)}{\sqrt{(\alpha - \varepsilon_-)^2 + \gamma^2}} \tag{3.74}$$

Substituting Equation 3.74 back into Equation 3.46 yields

$$x_1 = \frac{-\gamma}{\sqrt{(\alpha - \varepsilon_-)^2 + \gamma^2}} \tag{3.75}$$

Therefore,

$$\mathbf{x}_-^{(1)} = \frac{1}{\sqrt{(\alpha - \varepsilon_-)^2 + \gamma^2}} \begin{pmatrix} -\gamma \\ \alpha - \varepsilon_- \end{pmatrix} \tag{3.76}$$

Check:

$$\tilde{\mathbf{C}}\mathbf{x}_{-}^{(1)} = \frac{1}{\sqrt{(\alpha-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha & \gamma\\ \gamma & \beta\end{pmatrix}\begin{pmatrix}-\gamma\\ \alpha-\varepsilon_{-}\end{pmatrix} \tag{377}$$

$$= \frac{1}{\sqrt{(\alpha-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}-\alpha\gamma+\alpha\gamma-\gamma\varepsilon_{-}\\ -\gamma^{2}+\alpha\beta-\beta\varepsilon_{-}\end{pmatrix} \tag{3.78}$$

$$= \frac{1}{\sqrt{(\alpha-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}-\gamma\varepsilon_{-}\\ \varepsilon_{+}\varepsilon_{-}-\beta\varepsilon_{-}\end{pmatrix} \tag{3.79}$$

$$= \frac{\varepsilon_{-}}{\sqrt{(\alpha-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}-\gamma\\ \varepsilon_{+}-\beta\end{pmatrix} \tag{3.80}$$

Now Equation 3.41 states

$$\varepsilon_{+}-\beta=\alpha-\varepsilon_{-}$$

Therefore,

$$\tilde{\mathbf{C}}\mathbf{x}_{-}^{(1)} = \varepsilon_{-}\mathbf{x}_{-}^{(1)}$$

showing that Equation 3.31 holds true.

In checking for numerical stability we note that when $\gamma \rightarrow \mathbf{0}$

$$\mathbf{x}_{-}^{(1)} \rightarrow \frac{1}{\alpha-\beta}\begin{pmatrix}0\\ \alpha-\beta\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix} \tag{3.81}$$

producing a stable solution.

(4) Solution for $\mathbf{x}_-^{(2)}$. The last solution presented is for $\mathbf{x}_-^{(2)}$, which corresponds to the solution of the second equation in Equation 3.45 when $\varepsilon = \varepsilon_-$

$$\gamma x_1 + (\beta - \varepsilon_-)x_2 = 0 \tag{3.82}$$

$$x_1 = -\frac{\beta - \varepsilon_-}{\gamma} x_2 \tag{3.83}$$

The second equation needed to solve this system, as in the previous solutions, is supplied by normalizing the eigenvectors to unit length using Equation 3.46. By substitution we see

$$\left[\frac{(\beta - \varepsilon_-)^2}{\gamma^2} + 1\right] x_2^2 = 1 \tag{3.84}$$

Solving for x_2 we get

$$x_2^2 = \frac{\gamma^2}{(\beta - \varepsilon_-)^2 + \gamma^2} \tag{3.85}$$

$$x_2 = \frac{\gamma}{\sqrt{(\beta - \varepsilon_-)^2 + \gamma^2}} \tag{3.86}$$

Substituting Equation 3.86 back into Equation 3.46 yields

$$x_1 = \frac{\varepsilon_- - \beta}{\sqrt{(\beta - \varepsilon_-)^2 + \gamma^2}} \tag{3.87}$$

Therefore,

$$\mathbf{x}_-^{(2)} = \frac{1}{\sqrt{(\beta - \varepsilon_-)^2 + \gamma^2}} \begin{pmatrix} \varepsilon_- - \beta \\ \gamma \end{pmatrix} \tag{3.88}$$

Check:

$$\tilde{\mathbf{C}}\mathbf{x}_{-}^{(2)} = \frac{1}{\sqrt{(\beta-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha & \gamma \\ \gamma & \beta\end{pmatrix}\begin{pmatrix}\varepsilon_{-}-\beta \\ \gamma\end{pmatrix} \tag{3.89}$$

$$= \frac{1}{\sqrt{(\beta-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha\varepsilon_{-}-\alpha\beta+\gamma^{2} \\ \gamma\varepsilon_{-}-\beta\gamma+\beta\gamma\end{pmatrix} \tag{3.90}$$

$$= \frac{1}{\sqrt{(\beta-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha\varepsilon_{-}-\varepsilon_{+}\varepsilon_{-} \\ \gamma\varepsilon_{-}\end{pmatrix} \tag{3.91}$$

$$= \frac{\varepsilon_{-}}{\sqrt{(\beta-\varepsilon_{-})^{2}+\gamma^{2}}}\begin{pmatrix}\alpha-\varepsilon_{+} \\ \gamma\end{pmatrix} \tag{3.92}$$

Using Equation 3.41

$$\alpha-\varepsilon_{+}=\varepsilon_{-}-\beta$$

in Equation 3.92 gives

$$\tilde{\mathbf{C}}\mathbf{x}_{-}^{(2)}=\varepsilon_{-}\mathbf{x}_{-}^{(2)}$$

showing that Equation 3.31 holds true.

Checking for numerical stability we note that when $\gamma\rightarrow\mathbf{0}$ we obtain $\frac{0}{0}$ for the components of $\mathbf{x}_{-}^{(2)}$, therefore, $\mathbf{x}_{-}^{(2)}$ is not numerically stable as $\gamma\rightarrow\mathbf{0}$.

The eigenvalue ε_{-} like the eigenvalue ε_{+}, produced one computationally stable eigenvector. Note that $\varepsilon_{-}>0$ and $\varepsilon_{+}>0$ so both values are positive. The equations to be used in solving for the eigenvector matrix **G** will be: Equation 3.64 which corresponded to

the solution of the second equation in Equation3.45 when $\varepsilon = \varepsilon_+$, and Equation 3.76 which corresponds to the solution of the first equation of Equation 3.45 with $\varepsilon = \varepsilon_-$.

d. Eigenvector Matrix

This section will develop the solution for eigenvector matrix **G** and verify it by producing the diagonalized eigenvalue matrix Λ. The eigenvector matrix **G** is the matrix whose columns are the eigenvectors $\mathbf{x}_+^{(2)}$ and $\mathbf{x}_-^{(1)}$ (Recall these are the computationally stable eigenvector solutions found in sub-section c. of this chapter). In order to simplify the construction of **G,** some algebraic manipulation of the vectors is required. Consider Equation 3.64. Since $\varepsilon_+ - \beta = \alpha - \varepsilon_-$ (Identity Equation 3.41) we have

$$\mathbf{x}_+^{(2)} = \frac{1}{\sqrt{(\alpha - \varepsilon_-)(\varepsilon_+ - \beta) + \gamma^2}} \begin{pmatrix} \varepsilon_+ - \beta \\ \gamma \end{pmatrix}. \tag{3.93}$$

Now consider Equation 3.76. Again, since $\alpha - \varepsilon_- = \varepsilon_+ - \beta$ we can say that

$$\mathbf{x}_-^{(1)} = \frac{1}{\sqrt{(\alpha - \varepsilon_-)(\varepsilon_+ - \beta) + \gamma^2}} \begin{pmatrix} -\gamma \\ \alpha - \varepsilon_- \end{pmatrix}. \tag{3.94}$$

By combining Equation 3.93 and Equation 3.94 the eigenvector matrix **G** is

$$\mathbf{G} = \frac{1}{\sqrt{(\alpha - \varepsilon_-)(\varepsilon_+ - \beta) + \gamma^2}} \begin{pmatrix} \varepsilon_+ - \beta & -\gamma \\ \gamma & \alpha - \varepsilon_- \end{pmatrix}. \tag{3.95}$$

The determinant (difference of the products of the numbers in the two diagonals) of **G** is

$$\det(\mathbf{G}) = \frac{1}{\sqrt{(\alpha - \varepsilon_-)(\varepsilon_+ - \beta) + \gamma^2}} \left\{(\varepsilon_+ - \beta)(\alpha - \varepsilon_-) + \gamma^2\right\}, \tag{3.96}$$

which is equal to one, and the inverse of **G** is

$$\mathbf{G}^{-1} = \frac{1}{\sqrt{(\alpha-\varepsilon_-)(\varepsilon_+-\beta)+\gamma^2}}\begin{pmatrix} \alpha-\varepsilon_- & \gamma \\ -\gamma & \varepsilon_+-\beta \end{pmatrix}, \tag{3.97}$$

The diagonalized eigenvalue matrix Λ(the diagonal matrix with the eigenvalues ε_- and ε_+ as the diagonal components) is

$$\mathbf{G}^{-1}\tilde{\mathbf{C}}\mathbf{G} = \frac{1}{(\alpha-\varepsilon_-)(\varepsilon_+-\beta)+\gamma^2}\begin{pmatrix} \varepsilon_+-\beta & -\gamma \\ \gamma & \alpha-\varepsilon_- \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ -\gamma & \beta \end{pmatrix}\begin{pmatrix} \alpha-\varepsilon_- & \gamma \\ -\gamma & \varepsilon_+-\beta \end{pmatrix} \tag{3.98}$$

$$= \frac{1}{(\alpha-\varepsilon_-)(\varepsilon_+-\beta)+\gamma^2}\begin{pmatrix} \alpha-\varepsilon_- & \gamma \\ -\gamma & \varepsilon_+-\beta \end{pmatrix}\begin{pmatrix} \alpha\varepsilon_+-\alpha\beta+\gamma^2 & -\alpha\gamma+\alpha\gamma-\gamma\varepsilon_- \\ \gamma\varepsilon_+-\beta\gamma+\beta\gamma & -\gamma^2+\alpha\beta-\beta\varepsilon_- \end{pmatrix} \tag{3.99}$$

$$= \frac{1}{(\alpha-\varepsilon_-)(\varepsilon_+-\beta)+\gamma^2}\begin{pmatrix} \alpha-\varepsilon_- & \gamma \\ -\gamma & \varepsilon_+-\beta \end{pmatrix}\begin{pmatrix} \alpha\varepsilon_+-\varepsilon_+\varepsilon_- & -\gamma\varepsilon_- \\ \gamma\varepsilon_+ & \varepsilon_+\varepsilon_- -\beta\varepsilon_- \end{pmatrix} \tag{3.100}$$

$$= \frac{1}{(\alpha-\varepsilon_-)(\varepsilon_+-\beta)+\gamma^2}\begin{pmatrix} \alpha-\varepsilon_- & \gamma \\ -\gamma & \varepsilon_+-\beta \end{pmatrix}\begin{pmatrix} \varepsilon_+(\alpha-\varepsilon_-) & -\gamma\varepsilon_- \\ \gamma\varepsilon_+ & \varepsilon_-(\varepsilon_+-\beta) \end{pmatrix} \tag{3.101}$$

$$= \frac{1}{(\alpha-\varepsilon_-)(\varepsilon_+-\beta)+\gamma^2}\begin{pmatrix} \varepsilon_+(\alpha-\varepsilon_-)^2+\gamma^2\varepsilon_+ & -\gamma\varepsilon_-(\alpha-\varepsilon_-)+\gamma\varepsilon_-(\varepsilon_+-\beta) \\ -\gamma\varepsilon_+(\alpha-\varepsilon_-)+\gamma\varepsilon_+(\varepsilon_+-\beta) & \gamma^2\varepsilon_- +\varepsilon_-(\varepsilon_+-\beta)^2 \end{pmatrix} \tag{3102}$$

and by using $\alpha-\varepsilon_- = \varepsilon_+-\beta$ (Identity Equation 3.41) this becomes

$$\Lambda = \begin{pmatrix} \varepsilon_+ & 0 \\ 0 & \varepsilon_- \end{pmatrix} \tag{3103}$$

With this check it is reasonable to assume that Equation 3.95 is in fact the eigenvector matrix.

4. Variable Relationships

In the preceding text there have been two sets of relationships established. In Selection of Eigenvalues and Eigenvectors (Chapter III.A.2) a relationship between the

elements α, β and γ of the scaled covariance matrix $\tilde{\mathbf{C}}$ and the eigenvalues ε_+ , ε_- was established. A second relationship between the eigenvalues and the data parameters a, b, θ and χ^2 was detailed in Error Ellipse Relationship to the Bivariate Normal Distribution (Chapter II.A). The following section uses these relationships to develop $\tilde{\mathbf{C}}$ in terms of a, b, θ and χ^2.

The eigenvector from the positive root contains information relative to the direction of the semi major axis of the ellipse. Of the two positive root solutions, it was determined that the second was the more numerically stable. Therefore, the x, y components of the eigenvector $\mathbf{x}_+^{(2)}$ can be used to determine θ as illustrated in Figure 3.3.

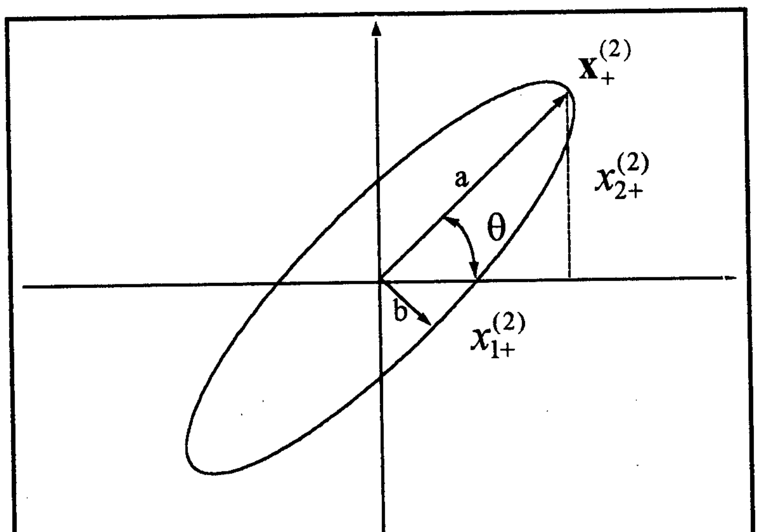


Figure 3.3 ELLIPSE ORIENTATION

From Equation 3.64 comes the following

$$\tan\theta = \frac{\mathrm{x}_{2+}^{(2)}}{\mathrm{x}_{1+}^{(2)}} = \frac{\gamma}{\varepsilon_+ - \beta} \tag{3.104}$$

The semi-axes for the ellipse corresponding to a probability level p are related to the eigenvalues ε_+ and ε_-. By Equations 2.22 and 2.23 we see that

$$a = \sqrt{\chi^2 \varepsilon_+} \quad \Rightarrow \quad \varepsilon_+ = \frac{a^2}{\chi^2} \tag{3.105}$$

and

$$b = \sqrt{\chi^2 \varepsilon_-} \quad \Rightarrow \quad \varepsilon_- = \frac{b^2}{\chi^2} \tag{3.106}$$

where χ^2 is the two-degree of freedom value of corresponding to probability level p. Manipulating Equation 3.104 we find

$$\varepsilon_+ - \beta = \gamma \cot\theta \tag{3.107}$$

which leads to the formula

$$\beta = \varepsilon_+ - \gamma \cot\theta . \tag{3.108}$$

From the Identity Equation 3.41 it follows that

$$\alpha = \varepsilon_+ + \varepsilon_- - \beta \tag{3.109}$$

$$= \varepsilon_+ + \varepsilon_- - \varepsilon_+ + \gamma \cot\theta \tag{3.110}$$

$$= \varepsilon_- + \gamma \cot\theta . \tag{3.111}$$

Substituting Equation 3.108 and Equation 3.111 into the Identity Equation 3.42 we find the second order polynomial

$$\varepsilon_+ \varepsilon_- = \left(\varepsilon_- + \gamma \cot\theta\right)\left(\varepsilon_+ - \gamma \cot\theta\right) - \gamma^2 \tag{3.112}$$

$$= \varepsilon_+ \varepsilon_- - \gamma\varepsilon_- \cot\theta + \gamma\varepsilon_+ \cot\theta - \gamma^2 \cot^2\theta - \gamma^2 \tag{3.113}$$

This can be rewritten as

$$-\gamma^2\left(\cot^2\theta+1\right)+\gamma\left(\varepsilon_+\cot\theta-\varepsilon_-\cot\theta\right)=0 \tag{3.114}$$

or

$$\gamma^2\left(\cot^2\theta+1\right)+\gamma\left(\varepsilon_-\cot\theta-\varepsilon_+\cot\theta\right)=0. \tag{3.115}$$

$$\gamma\left\{\gamma\left[\cot^2\theta+1\right]+\left[\varepsilon_-\cot\theta-\varepsilon_+\cot\theta\right]\right\}=0 \tag{3.116}$$

The non-zero solution of this equation is

$$\gamma=\frac{\varepsilon_+\cot\theta-\varepsilon_-\cot\theta}{\cot^2\theta+1} \tag{3.117}$$

$$=\frac{\cot\theta\left(\varepsilon_+-\varepsilon_-\right)}{\dfrac{1}{\sin^2\theta}} \tag{3.118}$$

$$=\sin^2\theta\cot\theta\left(\varepsilon_+-\varepsilon_-\right). \tag{3.119}$$

Seen in a more convenient form through application of the trigonometry identities as

$$\gamma=\sin\theta\cos\theta\left(\varepsilon_+-\varepsilon_-\right). \tag{3.120}$$

Recalling Equations 3.105 looking for. Back and 3.106 we see that Equation 3.120 is γ in terms of a, b, θ and χ^2, exactly what we were substituting for α and β we have

$$\alpha=\varepsilon_-+\gamma\cot\theta \tag{3.121}$$

$$=\varepsilon_-+\sin\theta\cos\theta\left(\varepsilon_+-\varepsilon_-\right)\cot\theta \tag{3.122}$$

$$=\varepsilon_-+\cos^2\theta\varepsilon_+-\cos^2\theta\varepsilon_- \tag{3.123}$$

leading to the useful form

$$\alpha = \varepsilon_- + \cos^2\theta(\varepsilon_+ - \varepsilon_-). \tag{3.124}$$

For β we get

$$\beta = \varepsilon_+ - \gamma\cot\theta \tag{3.125}$$

$$= \varepsilon_+ - \sin\theta\cos\theta(\varepsilon_+ - \varepsilon_-)\cot\theta \tag{3.126}$$

leading to the form we want

$$\beta = \varepsilon_+ - \cos^2\theta(\varepsilon_+ - \varepsilon_-). \tag{3.127}$$

Equations 3.120, 3.124 and 3.127 establish a direct relationship between the covariance matrix and the data parameters a, b, θ and χ^2.

B. N+1 COMBINATORIAL EFFECTS

The ability to combine multiple error ellipses, yielding the position vector and the associated covariance matrix is provided through Equation 2.51. This section is concerned with the effect of including an additional error ellipse to the existing solution, i.e. the N+1 case. In order to more clearly describe this problem, let us look at an example containing three ellipses. The question posed is will the solution for the case when all three observations are taken at once, as depicted in Figure 3.4,

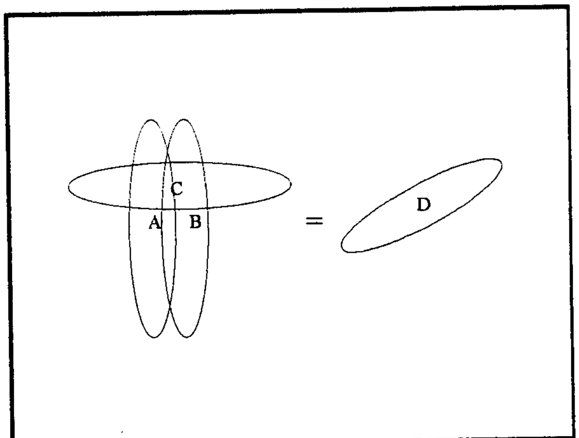


Figure 3.4 COMBINATION EXAMPLE (a)

be equal to the solution for the case in Figure 3.5, where two ellipsoids are initially processed, then a third is combined to that solution?

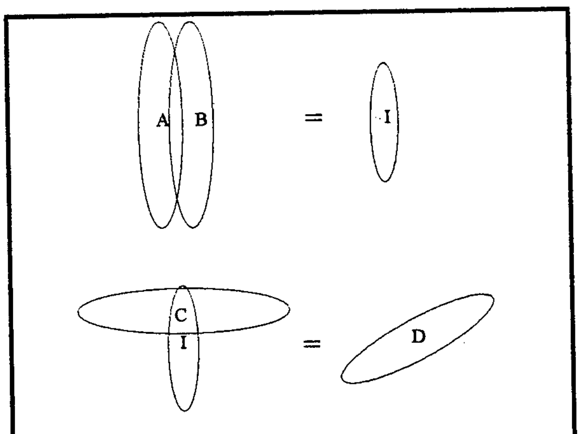


Figure 3.5 COMBINATION EXAMPLE (b)

Formally stated in Equation 3.128, we need to prove that the probability of the solution vector $\mathbf{x}$ given N+1 observations is equivalent to the probability of $\mathbf{x}$ given N observations given an additional observation.

$$P\left(P\left(\mathbf{x}\middle|\mathbf{o}_1\ldots\mathbf{o}_N\right)\middle|\mathbf{o}_{N+1}\right) = P\left(\mathbf{x}\middle|\mathbf{o}_1\ldots\mathbf{o}_{N+1}\right) \tag{3.128}$$

The optimal estimate for the combination of N observed error ellipses was derived in Chapter II. Substituting the scaled covariance matrix $\tilde{\mathbf{C}}$ into Equation 2.51 we see that

$$\mathbf{x} = \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\right)^{-1} \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i\right) = \tilde{\boldsymbol{C}}'_N \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i\right) \tag{3.129}$$

where

$$\tilde{\boldsymbol{C}}'_N = \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i\right)^{-1} \quad \Rightarrow \quad \tilde{\boldsymbol{C}}'^{-1}_N = \sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}. \tag{3.130}$$

$\tilde{\boldsymbol{C}}'$ is a new variable introduced to differentiate between the previously summed and the yet to be summed N+1 cases. The scaled covariance matrix for the solution of N+1 observations is

$$\tilde{\boldsymbol{C}}'^{-1}_{N+1} = \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1} + \tilde{\mathbf{C}}_{N+1}^{-1}\right) = \left(\tilde{\boldsymbol{C}}'^{-1}_N + \tilde{\mathbf{C}}_{N+1}^{-1}\right) \tag{3.131}$$

$$\tilde{\boldsymbol{C}}'_{N+1} = \left(\tilde{\boldsymbol{C}}'^{-1}_N + \tilde{\mathbf{C}}_{N+1}^{-1}\right)^{-1} \tag{3.132}$$

Now let's work through the N+1 solution

$$\mathbf{x}_{N+1} = \left(\sum_{i=1}^{N+1} \tilde{\mathbf{C}}_i^{-1}\right)^{-1} \left(\sum_{i=1}^{N+1} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i\right) \tag{3.133}$$

$$= \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1} + \tilde{\mathbf{C}}_{N+1}^{-1}\right)^{-1} \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i + \tilde{\mathbf{C}}_{N+1}^{-1}\mathbf{o}_{N+1}\right) \tag{3.134}$$

$$= \left(\tilde{\boldsymbol{C}}'^{-1}_N + \tilde{\mathbf{C}}_{N+1}^{-1}\right)^{-1} \left(\sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i + \tilde{\mathbf{C}}_{N+1}^{-1}\mathbf{o}_{N+1}\right) \tag{3.135}$$

$$\left(\tilde{\boldsymbol{C}}'^{-1}_N + \tilde{\mathbf{C}}_{N+1}^{-1}\right)\mathbf{x}_{N+1} = \sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i + \tilde{\mathbf{C}}_{N+1}^{-1}\mathbf{o}_{N+1} \tag{3.136}$$

$$\left(\tilde{\boldsymbol{C}}'^{-1}_N + \tilde{\mathbf{C}}_{N+1}^{-1}\right)\mathbf{x}_{N+1} = \tilde{\boldsymbol{C}}'^{-1}_N \tilde{\boldsymbol{C}}'_N \sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1}\mathbf{o}_i + \tilde{\mathbf{C}}_{N+1}^{-1}\mathbf{o}_{N+1} \tag{3.137}$$

where $\widetilde{\boldsymbol{C}}_N'^{-1}\widetilde{\boldsymbol{C}}_N' = \mathbf{I}$

$$\left(\widetilde{\boldsymbol{C}}_N'^{-1} + \widetilde{\mathbf{C}}_{N+1}^{-1}\right)\mathbf{x}_{N+1} = \widetilde{\boldsymbol{C}}_N'^{-1}\mathbf{x}_N + \widetilde{\mathbf{C}}_{N+1}^{-1}\mathbf{o}_{N+1} \tag{3.138}$$

$$\mathbf{x}_{N+1} = \left(\widetilde{\boldsymbol{C}}_N'^{-1} + \widetilde{\mathbf{C}}_{N+1}^{-1}\right)^{-1}\left[\widetilde{\boldsymbol{C}}_N'^{-1}\mathbf{x}_N + \widetilde{\mathbf{C}}_{N+1}^{-1}\mathbf{o}_{N+1}\right] \tag{3.139}$$

$$= \widetilde{\boldsymbol{C}}_{N+1}'\left(\sum_{i=1}^{N+1}\widetilde{\mathbf{C}}_i^{-1}\mathbf{o}_i\right) \tag{3.140}$$

Which agrees with Equation 3.128 proving the N+1 case is valid. This proof is critical from a software engineering point of view. Consider the N+1 case not valid. A piece of software wanting to keep a track updated over some amount of time, would need to store the information from all prior observations in order to update it's location and associated error ellipse. Since the N+1 case is valid, only the most recent solution needs to be retained. Allowing all previous fixes to be discarded, freeing memory for other tasking.

C. ELLIPSE ROTATION

A change of axes will be required in order to place the error ellipses that are to be combined in an approximately orthogonal coordinate system. In the latitude/longitude coordinate system, as latitude increases the coordinate system becomes progressively distorted. The closest approximation to an orthogonal coordinate system within the latitude/longitude coordinate system is found by transforming the ellipse center to a coordinate system where ϕ=0, λ=0. Where ϕ is latitude and λ longitude. This coordinate system will be referred to as the prime or $\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta$ coordinate system.

The process will begin by defining the $\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta$ coordinate system axes in terms of ϕ and λ with respect to the original $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system. Figure 3.6 depicts all axes and angles relevant to the transformation between the two coordinate systems.

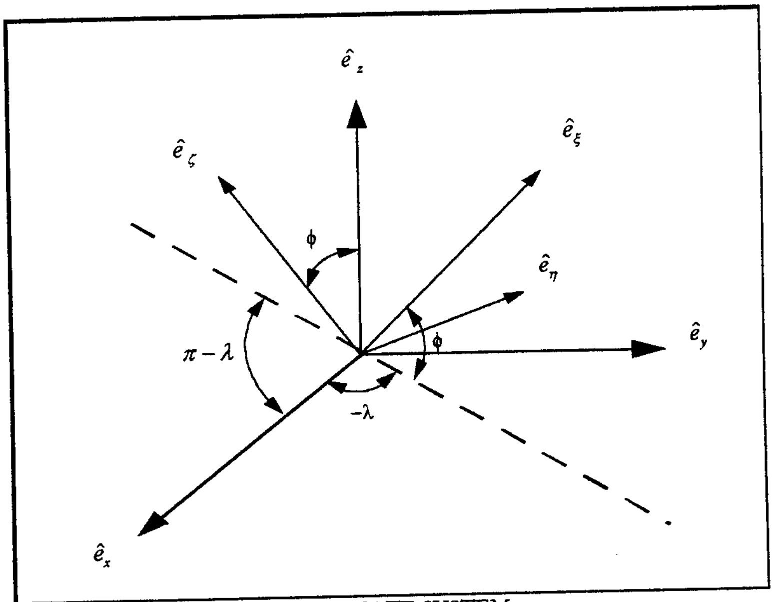


Figure 3.6 ROTATION COORDINATE SYSTEM

With the aid of Figure 3.6 the axis of the prime coordinate system $\hat{e}_\xi$, $\hat{e}_\eta$ and $\hat{e}_\zeta$ are defined as follows

$$\hat{e}_\xi = \cos(\lambda)\cos(\phi)\hat{e}_x + \sin(\lambda)\cos(\phi)\hat{e}_y + \sin(\phi)\hat{e}_z \tag{3.141}$$

$$\hat{e}_\zeta = \cos(\pi+\lambda)\cos\left(\frac{\pi}{2}-\phi\right)\hat{e}_x + \sin(\lambda+\pi)\cos\left(\frac{\pi}{2}-\phi\right)\hat{e}_y + \sin\left(\frac{\pi}{2}-\phi\right)\hat{e}_z \tag{3.142}$$

$$= -\cos(\lambda)\sin(\phi)\hat{e}_x - \sin(\lambda)\sin(\phi)\hat{e}_y + \cos(\phi)\hat{e}_z \tag{3.143}$$

As a simple check we verify that $\hat{e}_\xi$ and $\hat{e}_\zeta$ are orthogonal by confirming their dot product is zero.

$$\hat{e}_\xi \cdot \hat{e}_\zeta = -\cos^2(\lambda)\sin(\phi)\cos(\phi) - \sin^2(\lambda)\sin(\phi)\cos(\phi) + \sin(\phi)\cos(\phi) \tag{3.144}$$

$$= -\left[\cos^2(\lambda) + \sin^2(\lambda)\right]\sin(\phi)\cos(\phi) + \sin(\phi)\cos(\phi) \tag{3.145}$$

$$= 0$$

The third axis is easily found by the cross-product of the two previous directional vectors

$$\hat{e}_\eta = \hat{e}_\zeta \times \hat{e}_\xi \tag{3.146}$$

$$= -\sin(\lambda)\cos(\lambda)\sin(\phi)\cos(\phi)\hat{e}_z + \cos(\lambda)\sin^2(\phi)\hat{e}_y + \sin(\lambda)\cos(\lambda)\sin(\phi)\cos(\phi)\hat{e}_z$$

$$-\sin(\lambda)\sin^2(\phi)\hat{e}_x + \cos(\lambda)\cos^2(\phi)\hat{e}_y - \sin(\lambda)\cos^2(\phi)\hat{e}_x \tag{3.147}$$

$$= -\sin(\lambda)\hat{e}_x + \cos(\lambda)\hat{e}_y \tag{3.148}$$

To ensure thoroughness, the remaining combinations of axes are checked to be orthogonal

$$\hat{e}_\xi \cdot \hat{e}_\eta = -\sin(\lambda)\cos(\lambda)\cos(\phi) + \sin(\lambda)\cos(\lambda)\cos(\phi) = 0 \tag{3.149}$$

$$\hat{e}_\zeta \cdot \hat{e}_\eta = \sin(\lambda)\cos(\lambda)\sin(\phi) - \sin(\lambda)\cos(\lambda)\sin(\phi) = 0 \tag{3.150}$$

The unit vectors in the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system can be express as a relationship of the directional cosines relative to the $\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta$ coordinate system. In unit terms $\hat{i}_x \cdot \hat{i}_{x'} = \cos(\hat{i}_x, \hat{i}_{x'})$, which leads to the following

$$\hat{e}_x = (\hat{e}_x \cdot \hat{e}_\xi)\hat{e}_\xi + (\hat{e}_x \cdot \hat{e}_\eta)\hat{e}_\eta + (\hat{e}_x \cdot \hat{e}_\zeta)\hat{e}_\zeta \tag{3.151}$$

$$= \cos(\lambda)\cos(\phi)\hat{e}_\xi - \sin(\lambda)\hat{e}_\eta - \cos(\lambda)\sin(\phi)\hat{e}_\zeta \tag{3.152}$$

$$\hat{e}_y = (\hat{e}_y \cdot \hat{e}_\xi)\hat{e}_\xi + (\hat{e}_y \cdot \hat{e}_\eta)\hat{e}_\eta + (\hat{e}_y \cdot \hat{e}_\zeta)\hat{e}_\zeta \tag{3.153}$$

$$= \sin(\lambda)\cos(\phi)\hat{e}_\xi + \cos(\lambda)\hat{e}_\eta - \sin(\lambda)\sin(\phi)\hat{e}_\zeta \tag{3.154}$$

$$\hat{e}_z = (\hat{e}_z \cdot \hat{e}_\xi)\hat{e}_\xi + (\hat{e}_z \cdot \hat{e}_\eta)\hat{e}_\eta + (\hat{e}_z \cdot \hat{e}_\zeta)\hat{e}_\zeta \tag{3.155}$$

$$= \sin(\phi)\hat{e}_\xi + \cos(\phi)\hat{e}_\zeta \tag{3.156}$$

If we have coordinates $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system, the coordinates of the same point in the $\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta$ coordinate system are

$$\xi = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \hat{e}_\xi \cdot \hat{e}_x & \hat{e}_\xi \cdot \hat{e}_y & \hat{e}_\xi \cdot \hat{e}_z \\ \hat{e}_\eta \cdot \hat{e}_x & \hat{e}_\eta \cdot \hat{e}_y & \hat{e}_\eta \cdot \hat{e}_z \\ \hat{e}_\zeta \cdot \hat{e}_x & \hat{e}_\zeta \cdot \hat{e}_y & \hat{e}_\zeta \cdot \hat{e}_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3.157}$$

$$= \begin{pmatrix} \cos(\lambda)\cos(\phi) & \sin(\lambda)\cos(\phi) & \sin(\phi) \\ -\sin(\lambda) & \cos(\lambda) & 0 \\ -\cos(\lambda)\sin(\phi) & -\sin(\lambda)\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3.158}$$

The inverse transform is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \hat{e}_x \cdot \hat{e}_\xi & \hat{e}_x \cdot \hat{e}_\eta & \hat{e}_x \cdot \hat{e}_\zeta \\ \hat{e}_y \cdot \hat{e}_\xi & \hat{e}_y \cdot \hat{e}_\eta & \hat{e}_y \cdot \hat{e}_\zeta \\ \hat{e}_z \cdot \hat{e}_\xi & \hat{e}_z \cdot \hat{e}_\eta & \hat{e}_z \cdot \hat{e}_\zeta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{3.159}$$

$$= \begin{pmatrix} \cos(\lambda)\cos(\phi) & -\sin(\lambda) & -\cos(\lambda)\sin(\phi) \\ \sin(\lambda)\cos(\phi) & \cos(\lambda) & -\sin(\lambda)\sin(\phi) \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{3.160}$$

If we let ϕ_i, λ_i denote the latitude and longitude of a point in the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system. Then $\mathbf{x}_i$ can easily be expressed in terms ϕ_i and λ_i

$$\mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \begin{pmatrix} \cos(\phi_i)\cos(\lambda_i) \\ \cos(\phi_i)\sin(\lambda_i) \\ \sin(\phi_i) \end{pmatrix} \tag{3.161}$$

The coordinates of the point in the $\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta$ coordinate system are

$$\xi_i = \begin{pmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{pmatrix} = \begin{pmatrix} \cos(\lambda)\cos(\phi) & \sin(\lambda)\cos(\phi) & \sin(\phi) \\ -\sin(\lambda) & \cos(\lambda) & 0 \\ -\cos(\lambda)\sin(\phi) & -\sin(\lambda)\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\phi_i)\cos(\lambda_i) \\ \cos(\phi_i)\sin(\lambda_i) \\ \sin(\phi_i) \end{pmatrix} \quad (3.162)$$

If $\tilde{\phi}_i$, $\tilde{\lambda}_i$ denote the latitude and longitude of this point in the $\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta$ coordinate system, then

$$\xi_i = \begin{pmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{pmatrix} = \begin{pmatrix} \cos(\tilde{\phi}_i)\cos(\tilde{\lambda}_i) \\ \cos(\tilde{\phi}_i)\sin(\tilde{\lambda}_i) \\ \sin(\tilde{\phi}_i) \end{pmatrix}. \quad (3.163)$$

Equation 3.163 can be used to derive the latitude and longitude of the point in the prime coordinate system. Table 3.1 shows the range or value of the longitude of the point after it is rotated.

$$\tilde{\phi}_i = \sin^{-1}(\zeta_i) \qquad -\frac{\pi}{2} \le \tilde{\phi}_i \le \frac{\pi}{2} \quad (3.164)$$

$$\tilde{\lambda}_i = \tan^{-1}\left(\frac{\eta_i}{\xi_i}\right) \qquad -\pi \le \tilde{\lambda}_i \le \pi \quad (3.165)$$

Table 3.1 Ranges of Prime Longitude

If $\xi_i = 0, \eta_i = 0$	$\tilde{\lambda}_i \equiv 0$	If $\xi_i < 0, \eta_i > 0$	$\frac{\pi}{2} < \tilde{\lambda}_i < \pi$
If $\xi_i = 0, \eta_i > 0$	$\tilde{\lambda}_i = \frac{\pi}{2}$	If $\xi_i < 0, \eta_i < 0$	$-\pi < \tilde{\lambda}_i < -\frac{\pi}{2}$
If $\xi_i > 0, \eta_i > 0$	$0 < \tilde{\lambda}_i < \frac{\pi}{2}$	If $\xi_i < 0, \eta_i = 0$	$\tilde{\lambda}_i = \pi$
If $\xi_i > 0, \eta_i < 0$	$-\frac{\pi}{2} < \tilde{\lambda}_i < 0$	If $\xi_i > 0, \eta_i = 0$	$\tilde{\lambda}_i = 0$
If $\xi_i = 0, \eta_i < 0$	$\tilde{\lambda}_i = -\frac{\pi}{2}$		

In order to transpose the heading of an error ellipse between the two coordinate systems we must know the relationship between the semi-major direction vector and the north pole. The coordinates of the north pole in the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system are

$$\mathbf{x}_{np} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \phi_{np} = \frac{\pi}{2}, \lambda_{np} = 0 \tag{3.166}$$

The coordinates of this point in the $\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta$ coordinate system are found when Equation 3.158 is applied to the north pole from the $\hat{e}_x, \hat{e}_y, \hat{e}_z$ coordinate system

$$\xi_{np} = \begin{pmatrix} \xi_{np} \\ \eta_{np} \\ \zeta_{np} \end{pmatrix} = \begin{pmatrix} \cos(\lambda)\cos(\phi) & \sin(\lambda)\cos(\phi) & \sin(\phi) \\ -\sin(\lambda) & \cos(\lambda) & 0 \\ -\cos(\lambda)\sin(\phi) & -\sin(\lambda)\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.167}$$

$$= \begin{pmatrix} \sin(\phi) \\ 0 \\ \cos(\phi) \end{pmatrix} \tag{3.168}$$

Using Equation 3.168 we find the latitude and longitude of the north pole in the prime coordinate system

$$\tilde{\phi}_{np} = \sin^{-1}(\cos(\phi)) \quad , 0 \le \tilde{\phi}_{np} \le \frac{\pi}{2} \tag{3.169}$$

$$\tilde{\lambda}_{np} = \tan^{-1}\left(\frac{0}{\sin(\phi)}\right) \quad = \begin{cases} 0 \text{ if } \phi \ge 0 \\ \pi \text{ if } \phi < 0 \end{cases} \tag{3.170}$$

D. ELLIPSE HEADING RELATIVE TO NORTH POLE

The individual error ellipse headings relative to true north are changed during the axes transformation. Consider an ellipse at some arbitrary latitude, with a heading of true north. When the ellipse is transformed to the prime coordinate system, the heading is no

longer valid as depicted in Figure 3.7. This section will present the process which calculates this change in heading.

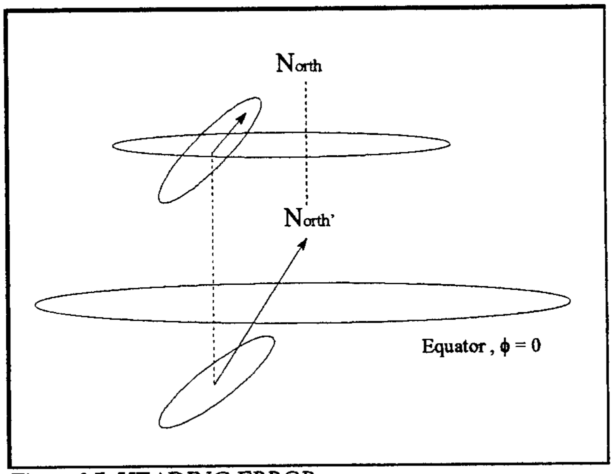


Figure 3.7 HEADING ERROR

The first step in this process is to determining each error ellipse heading relative to true north . Let two points have latitude and longitude ϕ_A, λ_A and ϕ_B, λ_B. We want to find the heading from A to B at point A measured relative to true north. The following pseudo code addresses the four trivial cases: one of the points is located on either of the poles, $\lambda_A = \lambda_B$ and $\lambda_A = \lambda_B \pm \pi$. First a point located on the north pole

$$\text{If } \phi_A = \frac{\pi}{2}, \phi_B \neq \frac{\pi}{2} \text{ then } \omega_{AB} = \pi$$

$$\text{If } \phi_B = \frac{\pi}{2}, \phi_A \neq \frac{\pi}{2} \text{ then } \omega_{AB} = 0$$

For the south pole

$$\text{If } \phi_A = -\frac{\pi}{2}, \phi_B \neq -\frac{\pi}{2} \text{ then } \omega_{AB} = 0$$

$$\text{If } \phi_B = -\frac{\pi}{2}, \phi_A \neq -\frac{\pi}{2} \text{ then } \omega_{AB} = \pi$$

Now the $\lambda_A = \lambda_B$ case

If $\lambda_A = \lambda_B$
  If $\phi_A \leq \phi_B$
    $\omega_{AB} = 0$

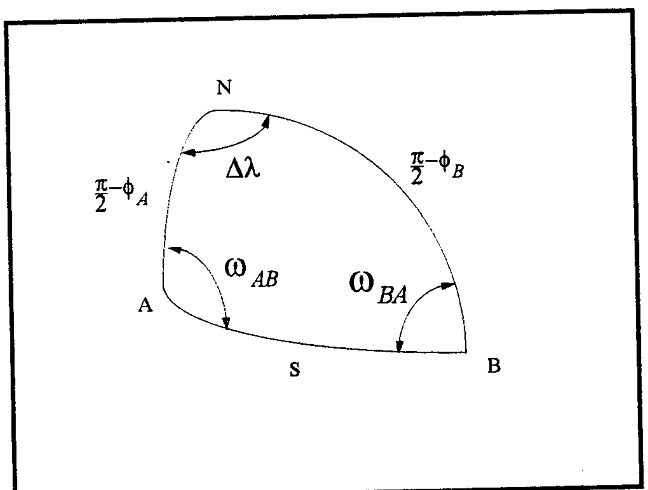
$$\text{Else If } \phi_A > \phi_B$$
$$\omega_{AB} = \pi$$

and the final trivial case, $\lambda_A = \lambda_B \pm \pi$

$$\begin{aligned}
&\text{If } \lambda_A = \lambda_B \pm \pi \\
&\quad \text{If } \phi_A \geq -\phi_B \\
&\qquad \omega_{AB} = 0 \\
&\quad \text{Else If } \phi_A < -\phi_B \\
&\qquad \omega_{AB} = \pi
\end{aligned}$$

If no trivial cases are valid apply Napier's analogies.

With the trivial cases out of the way we move on to examine the cases where Napier's analogies will need to be applied. Refer to Figure 3.8 for notation definitions used in the following problems. First consider the case where the longitude of point B is greater than that of point A



$$\Delta\lambda \equiv \lambda_B - \lambda_A$$
$$\Delta\lambda > 0\,;\; \Delta\lambda < \pi$$
$$\phi_A \neq \pm\frac{\pi}{2}\,;\; \phi_B \neq \pm\frac{\pi}{2}$$

Figure 3.8 HEADING BETWEEN POINTS B AND A

The points involved in this problem lie upon the surface of the earth. Taking this into account we must look to spherical trigonometry for a solution. Napier's analogies (Selby, 1967) provide the relationships from which this problem can be solved. Employed below we find that

$$\tan\left(\frac{1}{2}[\omega_{AB} - \omega_{BA}]\right) = \cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\sin\left(\frac{1}{2}\left[\frac{\pi}{2} - \phi_B - \frac{\pi}{2} + \phi_A\right]\right)}{\sin\left(\frac{1}{2}\left[\frac{\pi}{2} - \phi_B + \frac{\pi}{2} - \phi_A\right]\right)} \tag{3.171}$$

$$= \cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\sin\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\sin\left(\frac{1}{2}[\pi - \phi_A - \phi_B]\right)} \tag{3.172}$$

$$= \cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\sin\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\cos\left(\frac{1}{2}[\phi_A + \phi_B]\right)} \tag{3.173}$$

and that

$$\tan\left(\frac{1}{2}[\omega_{AB} + \omega_{BA}]\right) = \cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\cos\left(\frac{1}{2}\left[\frac{\pi}{2} - \phi_B - \frac{\pi}{2} + \phi_A\right]\right)}{\cos\left(\frac{1}{2}\left[\frac{\pi}{2} - \phi_B + \frac{\pi}{2} - \phi_A\right]\right)} \tag{3.174}$$

$$= \cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\cos\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\cos\left(\frac{1}{2}[\pi - \phi_A + \phi_B]\right)} \tag{3.175}$$

$$= \cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\cos\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\sin\left(\frac{1}{2}[\phi_A + \phi_B]\right)} \tag{3.176}$$

Equation 3.173 becomes

$$\frac{1}{2}[\omega_{AB} - \omega_{BA}] = \tan^{-1}\left\{\cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\sin\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\cos\left(\frac{1}{2}[\phi_A + \phi_B]\right)}\right\} \tag{3.177}$$

and Equation 3.176 is equivalent to

$$\frac{1}{2}[\omega_{AB} + \omega_{BA}] = \tan^{-1}\left\{\cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\cos\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\sin\left(\frac{1}{2}[\phi_A + \phi_B]\right)}\right\} \tag{3.178}$$

Therefore we can find ω_{AB} and ω_{BA} through the following two formulas

$$\omega_{AB} = \tan^{-1}\left\{\cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\cos\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\sin\left(\frac{1}{2}[\phi_A + \phi_B]\right)}\right\} + \tan^{-1}\left\{\cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\sin\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\cos\left(\frac{1}{2}[\phi_A + \phi_B]\right)}\right\} \tag{3.179}$$

$$\omega_{BA} = \tan^{-1}\left\{\cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\cos\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\sin\left(\frac{1}{2}[\phi_A + \phi_B]\right)}\right\} - \tan^{-1}\left\{\cot\left(\frac{1}{2}\Delta\lambda\right)\frac{\sin\left(\frac{1}{2}[\phi_A - \phi_B]\right)}{\cos\left(\frac{1}{2}[\phi_A + \phi_B]\right)}\right\} \tag{3.180}$$

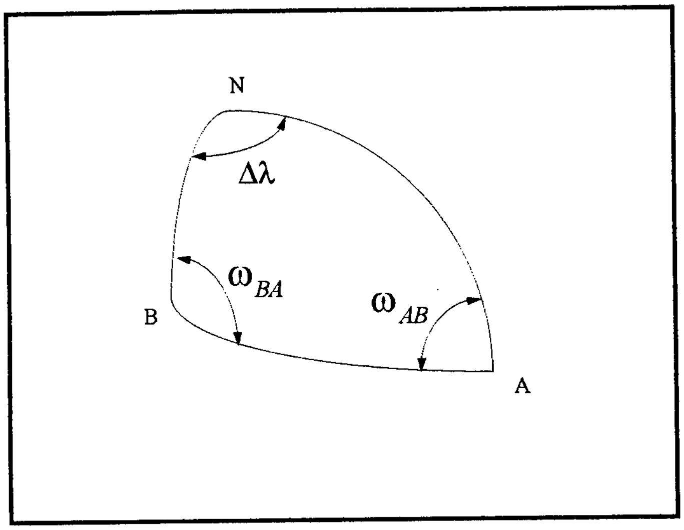
Putting these angles in terms of heading we find that the heading from point A to point B is

$$\text{Heading } A \rightarrow B = \omega_{AB} \tag{3.181}$$

and the heading from B to A is the mirror image of A to B

$$\text{Heading } B \rightarrow A = 2\pi - \omega_{BA} \tag{3.182}$$

Figure 3.9 illustrates the case when the longitude of A is larger than the longitude of B



$$\Delta\lambda \equiv \lambda_A - \lambda_B$$
$$\Delta\lambda > 0\;;\; \Delta\lambda < \pi$$
$$\phi_A \neq \pm\frac{\pi}{2}\;;\; \phi_B \neq \pm\frac{\pi}{2}$$

Figure 3.9 HEADING BETWEEN POINTS A AND B

Here the same formulas for ω_{AB} and ω_{BA} apply. The heading from point A to B is

$$\text{Heading } A \rightarrow B = 2\pi - \omega_{AB} \tag{3.183}$$

and the heading from B to A is the mirror image of A to B

$$\text{Heading } B \rightarrow A = \omega_{BA} \tag{3.184}$$

Now that we understand the how to obtain the heading between two points the next step is to apply this theory to our two coordinate systems. Let $\tilde{\omega}_i$ denote the heading from the center of each ellipse $\tilde{\phi}_i, \tilde{\lambda}_i$ to the north pole ξ_{np} in the $\xi\eta\zeta$-coordinate system. To find $\tilde{\omega}_i$, let $\tilde{\phi}_i, \tilde{\lambda}_i$ = point A and ξ_{np} = point B and execute the pseudo code in the beginning of this section. If θ_i is the heading of the ellipse at the point ϕ_i, λ_i in the xyz-coordinate system, then the heading of the ellipse at the point $\tilde{\phi}_i, \tilde{\lambda}_i$ in the $\xi\eta\zeta$-coordinate system is the sum of the original heading and the offset caused by rotation

$$\tilde{\theta}_i = \tilde{\omega}_i + \theta_i \tag{3.185}$$

IV. COMBINATION ALGORITHM

A. PROCESS OUTLINE

The Combination Algorithm will apply the theory and concepts from Chapters II and III to the real world environment. The input parameters for the combination algorithm will consist of a set of error ellipses with center ϕ_i , λ_i (i=1, ... , N), semi-major axis $\mathbf{A}_i$, semi-minor axis $\mathbf{B}_i$, heading θ_i and probability $\mathrm{p}_i (\Rightarrow \chi_i^2)$. The algorithm outlined below will process the N input ellipses and produce a solution ellipse that represents an area that has the specified probability that the target lie within it's bounds. Following is the Combination algorithm presented in a step by step manner.

1. Input a set of error ellipses with center ϕ_i , λ_i (i=1, ... , N), semi-major axis $\mathbf{A}_i$, semi-minor axis $\mathbf{B}_i$, heading θ_i and probability $\mathrm{p}_i (\Rightarrow \chi_i^2)$, where ϕ_i , λ_I refer to the latitude and longitude respectively. The semi-axes $\mathbf{A}_i$ and $\mathbf{B}_i$ are in an appropriate unit of length, θ_i in degrees and the desired probability level **p** refers to the confidence that the target lie within the generated solution ellipse.

2. Project semi-axes $\mathbf{A}_i$ and $\mathbf{B}_i$ onto a plane to obtain $\mathbf{a}_i$ and $\mathbf{b}_i$, where

$$\mathrm{a}_i = r_e \tan\left(\frac{\mathrm{A}_i}{r_e}\right) \tag{4.1}$$

$$\mathrm{b}_i = r_e \tan\left(\frac{\mathrm{B}_i}{r_e}\right) \tag{4.2}$$

Rationalization and development of Equation 4.1 and Equation 4.2 is presented in Chapter III.A.1, Scaled Covariance Matrix.

3. Find the eigenvalues corresponding to the semi-major axes $\mathbf{a}_i$, semi-minor axes $\mathbf{b}_i$ and the probability level **p** of each ellipse, where

$$\varepsilon_{+(i)} = \frac{a_i^2}{\chi_i^2} \tag{4.3}$$

$$\varepsilon_{-(i)} = \frac{b_i^2}{\chi_i^2} \tag{4.4}$$

Equation 4.3 and Equation 4.4 find their bases in Chapter II.A, Relationship of the Error Ellipse to a Bivariate Normal Distribution. The values of **p** for the χ^2 distributed function are given in Table 2.1.

4. Find the first approximation for the combined location of the center of the ellipses

$$x = \frac{1}{N}\sum_{i=1}^{N} x_i \quad = \quad \frac{r_e}{N}\sum_{i=1}^{N} \cos(\phi_i)\cos(\lambda_i) \tag{4.5}$$

$$y = \frac{1}{N}\sum_{i=1}^{N} y_i \quad = \quad \frac{r_e}{N}\sum_{i=1}^{N} \cos(\phi_i)\sin(\lambda_i) \tag{4.6}$$

$$z = \frac{1}{N}\sum_{i=1}^{N} z_i \quad = \quad \frac{r_e}{N}\sum_{i=1}^{N} \sin(\phi_i) \tag{4.7}$$

$$\phi = \sin^{-1}\left(\frac{z}{r_e}\right) \tag{4.8}$$

$$\lambda = \tan^{-1}\left(\frac{y}{x}\right) \tag{4.9}$$

The first part of this step finds the simple average of the (x,y,z) components of the individual error ellipses. Note that the information within the individual observation vectors $\mathbf{x}_i$, where $\mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$, should be retained for future use. These $\mathbf{x}_i$ vectors will be needed for Step 5. The second part of this step converts the averaged (x_i, y_i, z_i) components,

producing the latitude ϕ and longitude λ which is an approximation of the center of all the observed error ellipses.

5. Transform the location of the center of each error ellipse from xyz-coordinates to $\xi\eta\zeta$–coordinates (prime coordinates).

$$\begin{pmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{pmatrix} = \begin{pmatrix} \cos(\lambda)\cos(\phi) & \sin(\lambda)\cos(\phi) & \sin(\phi) \\ -\sin(\lambda) & \cos(\lambda) & 0 \\ -\cos(\lambda)\sin(\phi) & -\sin(\lambda)\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \mathrm{r_e}\cos(\phi_i)\cos(\lambda_i) \\ \mathrm{r_e}\cos(\phi_i)\sin(\lambda_i) \\ \mathrm{r_e}\sin(\phi_i) \end{pmatrix} \tag{4.10}$$

$$= \begin{pmatrix} \cos(\lambda)\cos(\phi) & \sin(\lambda)\cos(\phi) & \sin(\phi) \\ -\sin(\lambda) & \cos(\lambda) & 0 \\ -\cos(\lambda)\sin(\phi) & -\sin(\lambda)\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \tag{4.11}$$

$$\tilde{\phi}_i = \sin^{-1}\left(\frac{\zeta_i}{\mathrm{r_e}}\right) \tag{4.12}$$

$$\tilde{\lambda}_i = \tan^{-1}\left(\frac{\eta_i}{\xi_i}\right) \tag{4.13}$$

Note that $\mathbf{x}_i$ calculated in Step 4 does not need to be recalculated to obtain $\boldsymbol{\xi}_i$, where $\boldsymbol{\xi}_i = \begin{pmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{pmatrix}$. The transformation matrix used in Equation 4.10 is based on Equation 3.158 developed in Chapter III.C, Ellipse Rotation. During this step the first approximation coordinates ϕ,λ (Step 4) are the focus upon which the coordinate system is rotated as illustrated in Figure 4.1. The end product being the spatial relationship of each observation ϕ,λ to the first approximation ϕ_i,λ_i is maintained, but transformed to a coordinate system where $\phi = 0, \lambda = 0$. This transformation is required compensate for the difference in arc-length due to a fixed change in longitude as the latitude increases from zero.

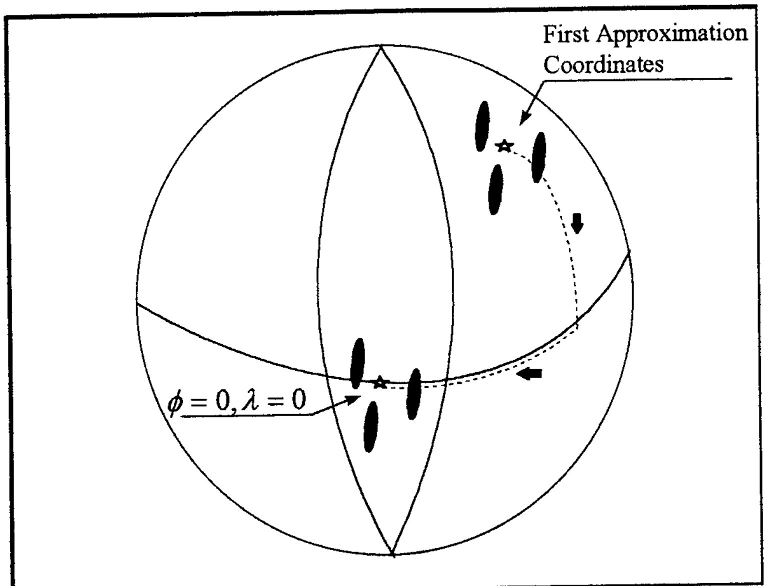


Figure 4.1 COORDINATE TRANSFORMATION

6. Find the coordinates of the north pole in $\xi\eta\zeta$–coordinates

$$\widetilde{\phi}_{np} = \sin^{-1}(\cos\phi) \tag{4.14}$$

$$\widetilde{\lambda}_{np} = \begin{cases} 0 & \text{if } \phi \geq 0 \\ \pi & \text{if } \phi < 0 \end{cases} \tag{4.15}$$

Location of the north pole in prime coordinates is needed to solve for the new ellipse heading after the coordinate rotation. Figure 4.2 shows the geometry involved in the pole rotation. Equations 4.14 and 4.15 were obtained form Equations 3.169 and 3.170.

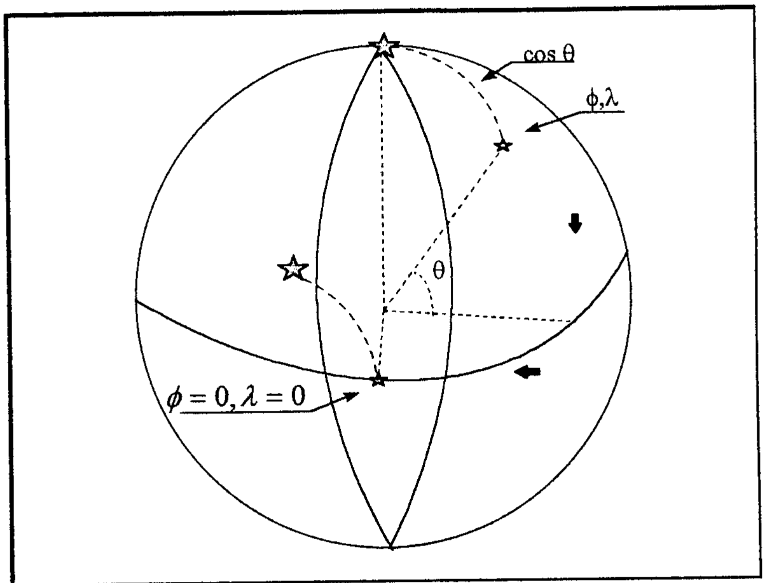


Figure 4.2 NORTH POLE ROTATION

7. Find the heading $\tilde{\omega}_i$ from the center of each error ellipse $\tilde{\phi}_i$, $\tilde{\lambda}_i$ to the north pole $\tilde{\phi}_{np}$, $\tilde{\lambda}_{np}$ in $\xi\eta\zeta$–coordinates.

The process for this step is detailed in Chapter III.D, Ellipse Heading Relative to North Pole. In execution, the first move is to examine the trivial cases $\tilde{\lambda}_i = \tilde{\lambda}_{np}$, $\tilde{\lambda}_i = \tilde{\lambda}_{np} + \pi$, and the case where one of the points is located on the north pole. In the likely event that none of the trivial case match, Napier's analogies must be utilized by employing Equation 3.179.

8..Find the heading of the error ellipses in $\xi\eta\zeta$–coordinates

$$\tilde{\theta}_i = \tilde{\omega}_i + \theta_i \tag{4.16}$$

The heading of the individual error ellipses in the prime coordinate system is the sum of the heading of each error ellipse $\tilde{\omega}_i$ (Step 7) and ellipse's heading in xyz-coordinates (Step 1).

9. Find the scaled covariance matrix corresponding to each error ellipse

$$\widetilde{\mathbf{C}}_i = \begin{pmatrix} \alpha_i & \gamma_i \\ \gamma_i & \beta_i \end{pmatrix} \tag{4.17}$$

$$\widetilde{\mathbf{C}}_i^{-1} = \frac{1}{\varepsilon_{+(i)}\varepsilon_{-(i)}} \begin{pmatrix} \beta_i & -\gamma_i \\ -\gamma_i & \alpha_i \end{pmatrix} \tag{4.18}$$

$$\alpha_i = \varepsilon_{-(i)} + \cos^2 \widetilde{\theta}_i \left(\varepsilon_{+(i)} - \varepsilon_{-(i)}\right) \tag{4.19}$$

$$\beta_i = \varepsilon_{+(i)} - \cos^2 \widetilde{\theta}_i \left(\varepsilon_{+(i)} - \varepsilon_{-(i)}\right) \tag{4.20}$$

$$\gamma_i = \sin \widetilde{\theta}_i \, \cos \widetilde{\theta}_i \left(\varepsilon_{+(i)} - \varepsilon_{-(i)}\right) \tag{4.21}$$

Along with the scaled covariance matrix $\widetilde{\mathbf{C}}_i$, Step 10 will require the individual ellipse scaled covariance matrix inverse $\widetilde{\mathbf{C}}_i^{-1}$ also be calculated. The elements of these matrices are derived in Chapter III.A.3, Variable Relationships.

10. Find the 2x2 scaled covariance matrix for the sum of the error ellipses

$$\widetilde{\mathbf{C}}^{-1} = \sum_{i=1}^{N} \widetilde{\mathbf{C}}_i^{-1} = \begin{pmatrix} \sigma & \upsilon \\ \upsilon & \tau \end{pmatrix} \tag{4.22}$$

$$\widetilde{\mathbf{C}} = \left(\widetilde{\mathbf{C}}^{-1}\right)^{-1} = \frac{1}{\sigma\tau - \upsilon^2} \begin{pmatrix} \tau & -\upsilon \\ -\upsilon & \sigma \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \tag{4.23}$$

Equation 4.22 is a result of the methodology outlined in Chapter II.C. The 2x2 scaled covariance matrix for the sum of the error ellipses $\widetilde{\mathbf{C}}$ produced in this step represents the solution ellipse that has the specified probability that the target lie within it's bounds. The location and heading of this solution ellipse has yet to be determined.

11. Convert the 2x2 scaled covariance matrix for the sum of the ellipses to the 3x3 matrix $\widetilde{\mathbf{C}}_{(\xi,\eta,\zeta)}$

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{r_e} & 0 \\ 0 & \frac{1}{r_e \cos\tilde{\phi}} \end{pmatrix} \tag{4.24}$$

$$\mathbf{C}_{(\tilde{\phi},\tilde{\lambda})} = \begin{pmatrix} \sigma_{\tilde{\phi}\tilde{\phi}} & \sigma_{\tilde{\phi}\tilde{\lambda}} \\ \sigma_{\tilde{\phi}\tilde{\lambda}} & \sigma_{\tilde{\lambda}\tilde{\lambda}} \end{pmatrix} = \left(\mathbf{S}^{-1}\right)^{\mathrm{T}} \tilde{\mathbf{C}}\, \mathbf{S}^{-1} \tag{4.25}$$

$$\mathbf{C}_{(\tilde{\phi},\tilde{\lambda},h)} = \begin{pmatrix} \sigma_{\tilde{\phi}\tilde{\phi}} & \sigma_{\tilde{\phi}\tilde{\lambda}} & 0 \\ \sigma_{\tilde{\phi}\tilde{\lambda}} & \sigma_{\tilde{\lambda}\tilde{\lambda}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.26}$$

$$\mathbf{T} = \begin{pmatrix} \frac{\partial \xi}{\partial \tilde{\phi}} & \frac{\partial \xi}{\partial \tilde{\lambda}} & \frac{\partial \xi}{\partial h} \\ \frac{\partial \eta}{\partial \tilde{\phi}} & \frac{\partial \eta}{\partial \tilde{\lambda}} & \frac{\partial \eta}{\partial h} \\ \frac{\partial \zeta}{\partial \tilde{\phi}} & \frac{\partial \zeta}{\partial \tilde{\lambda}} & \frac{\partial \zeta}{\partial h} \end{pmatrix} \tag{4.27}$$

$$\tilde{\mathbf{C}}_{(\xi,\eta,\zeta)} = \mathbf{T}\, \mathbf{C}_{(\tilde{\phi},\tilde{\lambda},h)} \mathbf{T}^{\mathrm{T}} \tag{4.28}$$

There are four tasks to accomplish prior to producing the 3x3 scaled covariance matrix for the sum of the ellipses $\tilde{\mathbf{C}}_{(\xi,\eta,\zeta)}$. First, the inverse of the scaling matrix defined in Equation 3.7 must be generated. Second, using the 2x2 scaled covariance matrix $\tilde{\mathbf{C}}$ from Step 10, calculate the unscaled covariance matrix in terms of latitude and longitude. Next, populate the 3x3 unscaled covariance matrix. This matrix $\mathbf{C}_{(\tilde{\phi},\tilde{\lambda},h)}$ is in terms of latitude, longitude and height. Lastly, the 3x3 scaling matrix $\mathbf{T}$ needs to be produced. $\mathbf{T}$ contains the partial derivatives of ξ, η and ζ with respect to $\hat{\phi}$, $\hat{\lambda}$ and h (Equations 3.19 thru 3.27). With those tasks complete $\tilde{\mathbf{C}}_{(\xi,\eta,\zeta)}$ can be calculated. This conversion process is described in detail in Chapter III.A.2.a 2x2 to 3x3 Covariance Matrix Conversion.

12.Find the weighted average location for the center of the error ellipse in $\xi\eta\zeta-$ coordinates

$$\boldsymbol{\xi} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \tilde{\mathbf{C}} \sum_{i=1}^{N} \tilde{\mathbf{C}}_i^{-1} \begin{pmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{pmatrix} \tag{4.29}$$

$$\tilde{\phi} = \sin^{-1}\left(\frac{\zeta}{\mathrm{r_e}}\right) \tag{4.30}$$

$$\tilde{\lambda} = \tan^{-1}\left(\frac{\eta}{\xi}\right) \tag{4.31}$$

The vector $\boldsymbol{\xi}$ represents the location of the weighted average of the N observations. The theory behind Equation 4.29 is presented in Chapter II.C, Methodology. The new latitude $\tilde{\phi}$ and longitude $\tilde{\lambda}$ are in essence refinements of the rotated first approximation. The level of refinement will be monitored once the solution is transformed back to the xyz-coordinate system. If convergence upon the true solution is not obtained, an iterative process will begin.

13. Find the eigenvalues of the combined covariance matrix.

$$\varepsilon_+ = \frac{1}{2}(\alpha+\beta) + \frac{1}{2}\left[(\alpha-\beta)^2 + 4\gamma^2\right]^{1/2} \tag{4.32}$$

$$\varepsilon_- = \frac{1}{2}(\alpha+\beta) - \frac{1}{2}\left[(\alpha-\beta)^2 + 4\gamma^2\right]^{1/2} \tag{4.33}$$

where α, β and γ are obtained from Equation 4.23.

The eigenvalues produced in this step are the key to extracting information about the semi-major axis, semi-minor axis and heading of the covariance matrix. Equation 4.32 and Equation 4.33 supply the real solutions to the characteristic polynomial of the covariance matrix discussed in Chapter III.A.3.a, Eigenvalues.

14. Find the heading of the error ellipse in the $\xi\eta\zeta$–coordinate system

$$\tilde{\theta} = \tan^{-1}\left(\frac{\gamma}{\varepsilon_{+} - \beta}\right) \tag{4.34}$$

The error ellipse heading is derived from the ξ and η components of the eigenvectors found in Section III.A.3.c, Eigenvectors. The development of Equation 4.34 can be found in Chapter III.A.4, Variable Relationships.

15. Find the heading $\tilde{\omega}$ from $\tilde{\phi}$, $\tilde{\lambda}$ to $\tilde{\phi}_{np}$, $\tilde{\lambda}_{np}$.

The same process found in Step 7 is utilized here.

16. Find the heading of the error ellipse in the xyz-coordinate system

$$\theta = \tilde{\theta} - \tilde{\omega} \tag{4.35}$$

The heading of the error ellipse in the xyz-coordinate system is found subtracting the heading offset due to rotation from the ellipses heading in the prime coordinate system. Note that Equation 4.35 agrees with Equation 4.16 from Step 8.

17. Find the coordinates of the weighted average location for the center of the error ellipses in xyz-coordinates

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos(\lambda)\cos(\phi) & -\sin(\lambda) & -\cos(\lambda)\sin(\phi) \\ \sin(\lambda)\cos(\phi) & \cos(\lambda) & -\sin(\lambda)\sin(\phi) \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{3.36}$$

$$\hat{\phi} = \sin^{-1}\left(\frac{\hat{z}}{\mathrm{r}_{e}}\right) \tag{4.37}$$

$$\hat{\lambda} = \tan^{-1}\left(\frac{\hat{y}}{\hat{x}}\right) \tag{4.38}$$

The transformation from prime coordinates to the xyz-coordinate system is developed in Chapter III.C, Ellipse Rotation. This step is the inverse of Step 5.

18. Check for convergence

If $|\phi - \hat{\phi}| < \kappa_\phi$ and $|\lambda - \hat{\lambda}| < \kappa_\lambda$ then
goto Step 19

else set

$x = \hat{x}$; $y = \hat{y}$; $z = \hat{z}$

$\phi = \hat{\phi}$; $\lambda = \hat{\lambda}$

and goto Step 5

In some cases the difference between the first approximation and the weighted average location my be significant. By setting values for κ_ϕ and κ_λ that are small an iterative process is established to fine tune the solution.

19. Calculate the semi-major and semi-minor axes of the combined error ellipse for the derived probability level

$$\mathrm{a} = \sqrt{\chi^2 \varepsilon_+} \tag{4.39}$$

$$\mathrm{b} = \sqrt{\chi^2 \varepsilon_-} \tag{4.40}$$

The semi-major and semi-minor axes of the combined error ellipse are found by utilizing the eigenvalues calculated in Step 14 and a selected chi-square number for Table II.1. Chapter II.A, Relationship of the Error Ellipse to A Bivariate Normal Distribution gives the origins of Equation 4.39 and Equation 4.40.

20. Project the semi-major and semi-minor axes from the plane to a spherical earth

$$\mathrm{A} = \mathrm{r_e} \tan^{-1}\left(\frac{\mathrm{a}}{\mathrm{r_e}}\right) \tag{4.41}$$

$$B = r_e \tan^{-1}\left(\frac{b}{r_e}\right) \tag{4.42}$$

The last step in the Combination Algorithm places solution ellipse back onto the spherical earth. This reverses Step 2 and supplies the last parameters of the solution ellipse which consists of latitude $\hat{\phi}$, longitude $\hat{\lambda}$, semi-major axes **A**, semi-minor axes **B**, and heading θ.

B. PROCESS FLOW CHART

Figure 4.3 provides a condensed view of the algorithm. For details of specific steps refer to Section A of this chapter.

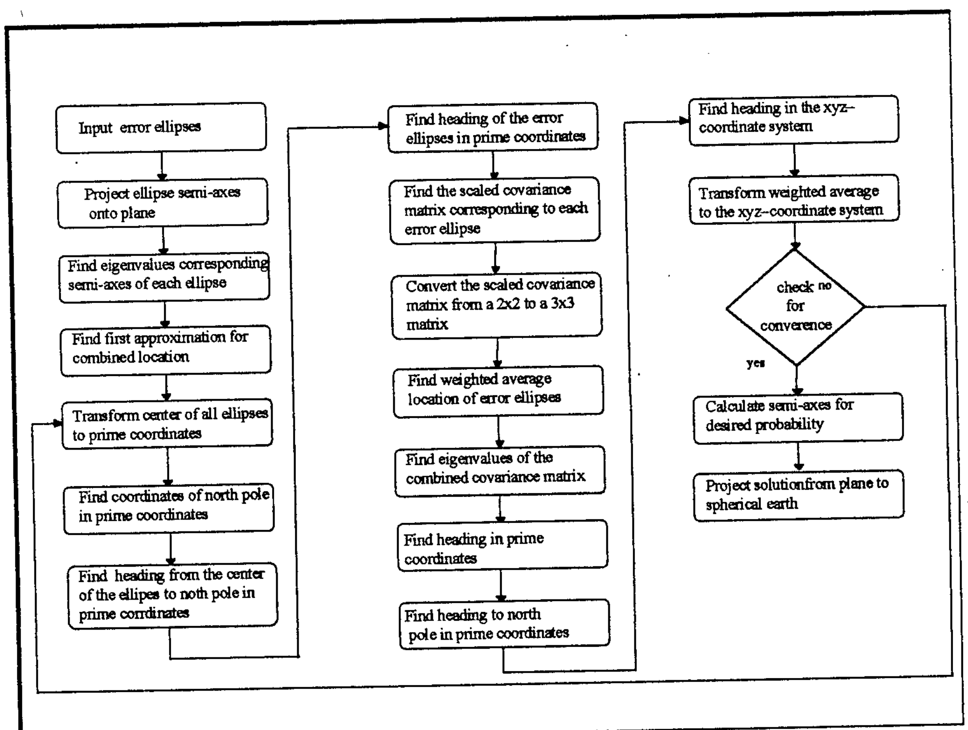


Figure 4.3 PROCESS FLOW CHART

V. SUMMARY & CONCLUSIONS

A. AREAS FOR FUTURE STUDY

One of the key assumptions made in the development of this algorithm was that the error ellipse data is derived from a bivariate normal distribution. Several transformations were made throughout the derivation to transform spherical data to a form that could be analyzed using a bivariate normal distribution. When the error ellipse is small ($a << r_e$) the probability distribution is highly localized and these transformations lead to a good approximation of a spherical probability distribution. The algorithm developed here will therefore be applicable to most error ellipses obtained from operational systems. Strictly speaking, however, we should have used a probability distribution more suited to spherical data, the Kent distribution (Fisher, Lewis, Embleton1987). While this distribution is more difficult to work with, it would be valid for error ellipses of all sizes. This issue should be examined more fully in the future.

There are three remaining areas relevant to this thesis that are available for future research. The first and foremost is to design and implement a test and validation program. The validation program would provide a means to quantitatively verify the theory put forth in this research. The project would entail producing computer code to implement the Combination Algorithm. Real world data sets are available to compare against output that would be generated by the program.

Broadening the Combination Algorithm to include multiple source data is the next potential research area. The ability to correlate observations from multiple platforms would greatly enhance our national warfighting capability. Decision makers would no longer need to mentally extrapolate a solution from numerous observations of the same target. The generation of this composite geolocation would reduce total decision cycle time, ultimately leading to increased force effectiveness. With an effective combination algorithm in place the need to transmit multiple geolocations of the same target would be eliminated, reducing fleet bandwidth requirements. By combining observations of the various platforms both bandwidth and data storage requirements are reduced. The reduction would be a result of

two factors: (1) collection platforms will combine same source observations reducing the amount of data to be transmitted, (2) once the collection platform solutions are received and combined into a composite solution the data received from the collection platforms can be discarded. The problem of including multiple source data could be approached by expanding the observation vector to include system dependent biases such as atmospheric delays and refraction.

The third area to consider for future research is the correlation of observations obtained of a moving target. Expanding the domain of possible targets from fixed site to include mobile platforms would significantly reduce the information presented to a decision maker. By expanding the number of platforms available for data reduction through combination, bandwidth and data storage resources are conserved. The inclusion of moving targets might be accomplished by sampling and processing a discrete number of hits over a large number of observations. Those interested in pursuing any one of these area can contact NRaD Code 841 for information on current research.

Areas of interest observed during initial attempts to implement the Combination Algorithm revolved around data structure design. Because the vast majority of operations to be performed in the algorithm involve matrix arithmetic, a language designed to for that purpose such as MATLAB is recommended. If C++ is to be used, it is recommended that an established toolbox of data objects and algorithms such as M++ by Dayd Software be utilized. Because of the large number of multi-dimensioned arrays required to code this algorithm a large memory model will be required. In the event that a smaller memory model must be used, the number of observations the program will be able to process will be severely restricted. To further reduce the strain on memory resources dynamic allocation of objects should be incorporated.

B. CONCLUSION

This thesis has set the theoretical basis and developed an algorithm for optimally combining multiple small spherical error ellipses into a single error ellipse. This algorithm is applicable to most of the error ellipses produced by operational systems. When this algorithm is validated it could have a major impact on how geolocation data is presented to the warfighter. In the future, as the electronic density of the battle field increases, the need for fusion of geolocation data will become increasingly apparent. This work can be considered the first step toward providing the warfighter with a fused multi-sensor geolocation solution.

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